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INITIAL SEGMENTS OF THE DEGREES OF SIZE $\aleph_1$

BY

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ABSTRACT

We settle a series of questions first raised by Yates at the Jerusalem (1968) Colloquium on Mathematical Logic by characterizing the initial segments of the degrees of unsolvability of size $\aleph_1$: Every upper semi-lattice of size $\aleph_1$ with zero, in which every element has at most countably many predecessors, is isomorphic to an initial segment of the Turing degrees.

Introduction

The study of initial segments (or equivalently the ideals) of the Turing degrees, $\mathscr{D}$, has been a major concern of Recursion Theory since Post [13] and Kleene and Post [6] began the systematic investigation of the structure of the degrees under T-reducibility. The first result was the existence of a minimal degree proven by Spector [21] to answer the question raised in Kleene and Post [6]. Since that time there has been a long sequence of questions, conjectures and theorems by many people elucidating more and more of the possible initial segments of $\mathscr{D}$. We cite just a few of the key steps: Countable linear orderings, Hugill [4]; countable distributive lattices, Lachlan [7]; all finite lattices, Lerman [9]; all countable upper semi-lattices, Lachlan and Lebeuf [8]. The techniques developed in these papers have been applied to many other degree structures from 1–1 degrees to degrees of constructibility. Indeed their analogs in set theory (perfect forcing or Sacks forcing) have had applications beyond those to degree structures. Within recursion theory the results have come to play a key role in

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the analysis of the global structure of $\mathscr{D}$ and so in answering much more general questions. Perhaps the first such application was by Feiner [2] who used the results on linear orderings to refute the strong homogeneity conjecture. Lachlan's [7] result of course gave the undecidability of the theory of $\mathscr{D}$ while it or other initial segments results played a key role in all the more recent work on the global structure of $\mathscr{D}$ as in Simpson [20] or Nerode and Shore [11] which characterizes the degree of Th($\mathscr{D}$) as that of true second order arithmetic. Other applications include the refutation of the homogeneity conjecture in Shore [18], restrictions on possible automorphisms of $\mathscr{D}$ in Nerode and Shore [12] and various definability results in $\mathscr{D}$ as, for example, in Jockusch and Shore [5]. In another direction Lerman's result on finite lattices was the key ingredient in the proof of the decidability of the two quantifier theory of $\mathscr{D}$ (Shore [17] and Lerman [10]). A reasonable survey can be found in Shore [19].

Now all of these results have dealt with just the countable initial segments of $\mathscr{D}$. Although there were some early isolated results on the uncountable ones (e.g., Thomason [22]) they remained largely mysterious. The problem as to what they might be was first raised in Yates [23] in a series of questions about the initial segments of $\mathscr{D}$ of size $\aleph_1$. At the time there was some feeling that the answers might be independent of ZFC and a consistency result for an initial segment of type ω_1 was pointed out. As it turned out, he was both right and wrong. He was right in that the independence phenomenon was lurking in the initial segments problem but wrong in that it does not appear with ones of size $\aleph_1$: Groszek and Slaman [3] prove that it is consistent (relative to the consistency of ZFC) that the continuum is large (e.g., $2^{\aleph_0} > \aleph_2$) and there is an upper semi-lattice (u.s.l.) [with 0 and the countable predecessor property] of size $\aleph_2$ (and so $< 2^{\aleph_0}$) which is not isomorphic to an initial segment of $\mathscr{D}$. In this paper we will give positive answers to the entire sequence of questions of Yates [23, §6] by proving (in ZFC) that every u.s.l. of size $\aleph_1$ with 0 and the countable predecessor property is isomorphic to an initial segment of $\mathscr{D}$. (Of course as $\mathscr{D}$ has a least element **0** and every degree has at most countably many predecessors, every initial segment of $\mathscr{D}$ of size $\aleph_1$ must have these properties.) We also describe the minor additions needed to get the result for *tt* and *wtt* degrees as well.

We should mention that some partial results along these lines (for ω_1 and then distributive lattices) were announced in Rubin [14] and [15] but no write up ever appeared and we do not know what were his intended constructions. The motivation for our basic approach to iterating the initial segment construction into the transfinite comes from a forcing argument in Shelah [16].

As in Lerman [10] one should, as Sacks first suggested, view the initial segment constructions in recursion theory as forcing arguments where conditions are recursive perfect trees and generic objects are those which meet certain specified collections of dense sets. Suppose that $\mathcal{P}$ is the appropriate notion of forcing for embedding a countable u.s.l. $\mathcal{L}$ as an initial segment of $\mathcal{D}$. A condition will consist of finitely many elements of $\mathcal{L}$, trees for each one and maps between them. One then specifies a collection of dense sets $\mathcal{C}$ such that any $\mathcal{C}$-generic filter $\mathcal{G}$ on $\mathcal{P}$ gives an isomorphism of $\mathcal{L}$ onto an initial segment of $\mathcal{D}$ by sending x to the degree of the branch of the tree associated with x determined by $\mathcal{G}$ ($G_x = \bigcup\{T_{p,x}(\phi) \mid P \in \mathcal{G}\}$ and $x \mapsto \deg G_x$). The problem now is how to extend $\mathcal{G}$ to a $\mathcal{C}$-generic filter $\mathcal{G}'$ on $\mathcal{P}'$, the notion of forcing for some $\mathcal{L}' \supseteq \mathcal{L}$, and to do this in an iterable way so as to be able to carry on through ω_1-many extensions. The idea of Shelah [16] is that one restricts $\mathcal{P}'$ to those conditions which are represented by conditions in $\mathcal{G}$ via some isomorphism. More precisely if $\phi : \mathcal{L} \to \mathcal{L}'$ is a partial (u.s.l.) isomorphism and $P \in \mathcal{G}$ then $P' = \phi(P)$ is an element of $\mathcal{P}'$ where P' is gotten from P by relabelling every element x as $\phi(x)$. To make sure that any $\mathcal{C}$-generic filter $\mathcal{G}'$ on $\mathcal{P}'$ extends $\mathcal{G}$, one requires that $\phi^{-1} \restriction \mathcal{L} = \text{id}$. If one can define $\mathcal{P}$ and $\mathcal{C}$ so that such an extension is always possible then one can follow a division of a given u.s.l. $\mathcal{L}^*$ of size $\aleph_1$ into countable sub u.s.l.'s, $\mathcal{L}^* = \bigcup_{\alpha<\omega_1} \mathcal{L}_\alpha$, to build a monotonic sequence of $\mathcal{C}$-generic filters $\mathcal{G}_\alpha$ for the appropriate notions of forcing $\mathcal{P}_\alpha$ such that $\mathcal{G}^* = \bigcup_{\alpha<\omega_1} \mathcal{G}_\alpha$ defines an isomorphism of $\mathcal{L}^*$ onto an initial segment of $\mathcal{D}$ the same way $\mathcal{G}$ did for the original countable $\mathcal{L}$.

We carry out this program for linear orderings in Section 1. First (Theorem 1.21) we give a fairly standard presentation of the countable case, basically in the style of Lachlan [7] as presented in Epstein [1] with a couple of minor modifications to pave the way for the extension process. We then proceed to the size $\aleph_1$ case. Of course the key problem is the choice of the appropriate dense sets (and the proof that they are dense) to permit the extension process to proceed. These are to be found in Definitions 1.22 and 1.23 and Lemmas 1.24–1.26 along with motivation for their precise form. Lemma 1.27 then carries out the inductive argument by showing that if $\mathcal{G}_\alpha$ is $\mathcal{C}$-generic for $\mathcal{P}_\alpha$ then the sets in $\mathcal{C}$ are dense in $\mathcal{P}_{\alpha+1}$ and so there is a $\mathcal{G}_{\alpha+1} \supseteq \mathcal{G}_\alpha$ $\mathcal{C}$-generic for $\mathcal{P}_{\alpha+1}$. As limit levels are essentially trivial ($\mathcal{G}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{G}_\alpha$ and $\mathcal{P}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{P}_\alpha$) this completes the proof for linear orderings, Theorem 1.29.

Unfortunately, the result for arbitrary u.s.l.'s is considerably more complicated than that for linear orderings (or even distributive lattices which much resemble linear orderings). Of course there are the severe extra complications even in the countable case when one gives up distributivity. These are presented in Section 2 in the style of Lerman [10] (again with some minor modifications to pave the way for the extension process) where we present Lachlan and Lebeuf's result for countable u.s.l.'s (Theorem 2.17). Much more, however, is needed in the general case than was done in Section 1 to carry the extension procedure into the transfinite. The bulk of the paper (Sections 3 and 4) is devoted to this problem.

There are two main points. The first is that only very special conditions P in $\mathcal{G}_\alpha$ can be used to represent ones P' in $\mathcal{P}_{\alpha+1}$ via an isomorphism ϕ. Roughly speaking $\phi^{-1}[L_{P'}]$ must be as free as possible over $\phi^{-1}[L_P \cap \mathcal{L}_\alpha]$. [$L_P$ is the finite u.s.l. whose elements are mentioned in P.] The precise definition is motivated and then presented in Definition 3.2. Various needed algebraic properties of such extensions are then established in Lemmas 3.3–3.6. We can then define (3.7) the notions of forcing $\mathcal{P}_\alpha$ modulo the correct choice of the class $\mathcal{C}$ of dense sets. Their definition is motivated and then given in Definitions 3.8 and 3.9. Assuming the density of these sets at the initial level the inductive argument is then given in Lemma 3.10.

What then remains is the demonstration of the density of the sets needed for the inductive argument. The proof is provided in Section 4. The key ingredient here is an extension of the u.s.l. representation theorems proved by Lerman [9] and Lachlan and Lebeuf [8] that exploits the special extension introduced in Section 3 to enable us to refine a nice representation of a given finite u.s.l. to one for a larger one containing two isomorphic copies of (some part of) the first in such a way that each one induces the same reduction procedures on the associated sets being constructed. This is Theorem 4.1.

We follow the style and notation of Lerman [10] as much as possible. We have, however, included all definitions dealing specifically with initial segments results. Section 1 is in fact self contained and can be read without previous knowledge of initial segments results. (One does need to know that $\{\phi_e^X\}$ is a list of all possible Turing reductions from X.) In Section 2, however, we have relied on Lerman's [10, chapter VII] embedding of finite lattices as initial segments in that we refer to that book for the proof of two key lemmas (2.14 and 2.16). Similarly we rely on his construction of suitable representations for finite lattices [10, appendix B, §2] in our proof of Theorem 4.1. Otherwise, the paper is self contained.

1. Linear orderings

Our goal in this section is to give a self-contained proof of our embedding theorem for the special case of linear orderings: Every linear ordering $\mathcal{L}^*$ of size

$\aleph_1$ with least element and the countable predecessor property (i.e., $\{y \mid y \leq x\}$ is countable for every $x \in \mathscr{L}^*$) is isomorphic to an initial segment of the Turing (*wtt* and *tt*) degrees. Although many of the problems encountered in the general case of arbitrary upper semi-lattices do not appear here the main idea motivating the construction can be seen relatively clearly.

We begin with a proof for countable orderings $\mathscr{L}$ which is then extended to uncountable ones. (See the discussion following Theorem 1.21 culminating with Definition 1.22 of the forcing notion and Definition 1.23 of the required dense sets for an explanation of this extension process.) Most of our notations and presentations are those of Lerman [10] although in the case of linear orderings almost all notions of lattice representations are suppressed in favor of an unstated representation within the recursive sets under inclusion as used, for example, in Lachlan [7] or Epstein [1]. (We, of course, must bring the representations and associated lattice tables out in full force in the general case.)

A more germane difference from Lerman [10] as well as other common presentations is that we cannot assume that $\mathscr{L}$ has a maximum element if we hope to eventually extend the embedding to one of an $\mathscr{L}^*$ of size $\aleph_1$. Thus we cannot work with a single master tree approximating such a maximum element but must have conditions with distinct trees T_i for each of the elements $i \in \mathscr{L}$ being approximated by the condition. The role of the congruence relations that dictate the decoding of the sets corresponding to other elements of $\mathscr{L}$ from the branch on the master tree is played by a (commutative) family of recursive maps sending branches of T_j to ones of T_i for i less than j in $\mathscr{L}$.

These ideas are embodied in Definitions 1.3 and 1.6 which should therefore be studied even by those familiar with Lerman [10]. Such a reader can then skim to the end of the proof of the embedding result for a countable $\mathscr{L}$ (Theorem 1.21). A reader familiar with some other proof of this result should go over all the definitions and statements of the lemmas to become familar with the notational setup. The proofs, however, are essentially standard. The only one even slightly out of the ordinary (because of our not assuming a maximum element) is Lemma 1.11 which is worth a look for that reason. Finally, for the reader who has never seen or no longer remembers any initial segments results (except perhaps the existence of a minimal degree) we have included all basic definitions and complete proofs.

DEFINITION 1.1. *Strings.* (a) $\mathscr{S}$ is the set of all strings σ, i.e., all finite sequences of natural numbers or more formally all maps $\sigma : n \to \omega$ for some $n \in \omega$.

(b) The *length* of a string σ, $\operatorname{lth}\sigma$, is its domain.

(c) We *order* strings by extension $\sigma \subseteq \tau$ iff $\forall n, m\ [\sigma(n) = m \rightarrow \tau(n) = m]$.

(d) For a given function $f : \omega \rightarrow [\omega]^{<\omega}$ we let $\mathcal{S}_f$ be the set of all *f-strings*, i.e., all σ such that $\forall x < \operatorname{lth}\sigma\ (\sigma(x) \in f(x))$ ($[\omega]^{<\omega}$ is the set of all finite subsets of ω). In particular if $f(x) \equiv p = \{0, 1, \ldots, p-1\}$ we call these *p-ary strings*, e.g., if $p = 2$ these are the *binary strings*.

Definition 1.2. *Trees*. Let $f : \omega \rightarrow [\omega]^{<\omega}$ be given.

(a) An *f-tree* is a map $T : \mathcal{S}_f \rightarrow \mathcal{S}_f$ such that $(\forall \sigma, \tau \in \mathcal{S}_f)[\sigma \subseteq \tau \Leftrightarrow T(\sigma) \subseteq T(\tau)]$.

(b) τ *is on* T iff $\exists \sigma\ [\tau = T(\sigma)]$.

(c) τ *is compatible with* T iff $\exists \sigma\ [\tau \subseteq T(\sigma)]$.

(d) h *is on* T iff $\forall \tau \subseteq h$ [τ is compatible with T]. In this situation we call h a *branch* of T. It is associated with a *path* g through T such that $h = T[g] = \bigcup_{\sigma \subset g} T(\sigma)$. $[T] = \{h \mid h \text{ is on } T\}$.

(e) T is *recursive* if it is recursive as a function.

(f) T is *uniform* if $(\forall n)$ $(\exists \{\rho_j \mid j \in f(n)\}$ of equal length)

$$(\forall \sigma \text{ of length } n)(\forall j \in f(n))[T(\sigma * j) = T(\sigma) * \rho_j].$$

(g) T^* is a *subtree* of T, $T^* \subseteq T$, iff $\operatorname{rg} T^* \subseteq \operatorname{rg} T$.

Note. One can specify an (f-) subtree T^* of an (f-) tree T by giving an (f-) tree S and setting $T^* = T \circ S$. Now if T and S are uniform so is T^*. One can in this case also specify T^* by induction on length σ by giving at level n for each $j \in f(n)$ the string ρ_j such that if $T^*(\sigma) = T(\tau)$, then $T^*(\sigma * j) = T(\tau * \rho_j)$. Thus, for example, if $T^* \subseteq T$ are both uniform then $(\forall n)(\exists m)(\forall \sigma$ of length $n)(\exists \tau$ of length $m)$ $(T^*(\sigma) = T(\tau))$.

For the rest of this section all strings will be binary and all tress will be binary uniform and recursive. As we identify a set with its characteristic function we will speak of a set G being on a tree T, determining a path through T, etc.

Let $\mathcal{L}$ be a given countable linear ordering (with least element 0) specified by $<$. We wish to define a *notion of forcing*, i.e., a partially ordered set $(\mathcal{P}, \leqq)$ and a class $\mathcal{C}$ of dense (i.e., downwardly cofinal) sets such that any $\mathcal{C}$-generic filter $\mathcal{G}$ specifies an embedding of $\mathcal{L}$ as an initial segment of the Turing degrees $\mathcal{D}$. [Recall that $\mathcal{G} \subseteq \mathcal{P}$ is *$\mathcal{C}$-generic* if

(i) $\forall P \in \mathcal{G}\ \forall Q \geqq P (Q \in \mathcal{G})$,

(ii) $\forall P, Q \in \mathcal{G}\ \exists R \in \mathcal{G}\ (R \leqq P\ \&\ R \leqq Q)$,

(iii) $\forall C \in \mathcal{C}\ (\mathcal{G} \cap C \neq \emptyset)$.]

The basic ingredients of our forcing conditions (elements of $\mathcal{P}$) will be trees T_i which we think of as approximating some G_i on T_i whose degree will be the image of $i \in \mathcal{L}$ under the hoped for embedding. We reflect the requirement that if $i \leqslant j$ then $G_i \leqq_T G_j$ by including recursive maps from $[T_j]$ to $[T_i]$ in our conditions. These maps will be specified by a recursive monotonic function f such that to see whether at level n the path C_i associated with G_i turns right ($C_i(n) = 0$) or left ($C_i(n) = 1$) one just asks which way the one C_j associated with G_j turns at level $f(n)$. Now if $\operatorname{rg} f = \omega$ (or is even cofinite) we could reverse this process to compute the path on T_j from the corresponding one on T_i. As we will want $G_j \not\leqq_T G_i$ if $j \not\leqslant i$ we consider only maps f with coinfinite range.

DEFINITION 1.3. *Projections.* (a) Let S and T be trees and f a recursive monotonic function with coinfinite range. We say that f *induces the recursive projection* $F: [T] \to [S]$ if $F(T[C]) = S[f^{-1}[C]]$ where $f^{-1}: \mathcal{S}_2 \to \mathcal{S}_2$ is given by $f^{-1}(\sigma)(n) = \sigma(f(n))$ and $f^{-1}[C] \equiv \bigcup_{\sigma \subseteq C} f^{-1}(\sigma)$.

Thus, for a given branch $T[C]$ following the path C through T, its image under F is the branch of S determined by the path of $f^{-1}[C]$ which turns right (left) at level n just if C does at level $f(n)$.

(b) In this situation we say that two strings σ and τ are *congruent* mod f, $\sigma \equiv_f \tau$, if $f^{-1}(\sigma) = f^{-1}(\tau)$. We say that level n of T is an *f-differentiating level* if for σ of length n $f^{-1}(\sigma * 0) \neq f^{-1}(\sigma * 1)$, i.e., $\sigma * 0 \not\equiv_f \sigma * 1$. Similarly if $T^* \subseteq T$ we say that a level n of T^* is *f-differentiating* (relative to T) if for σ of length n and $T^*(\sigma * r) = T(\tau_r)$, $\tau_0 \not\equiv_f \tau_1$.

(c) If f induces a projection $F: [T] \to [S]$ as above and $T^* \subseteq T$ has infinitely many f-differentiating levels then there is a natural subtree $S^* = F(T^*)$ which is the *projection of* T^*: Suppose we have defined S^* up to level n and for some σ of length n $S^*(\sigma) = S(\tau)$ and we have a ρ such that $f^{-1}(\rho) = \tau$ and $T(\rho) = T^*(\eta)$. Find the shortest $\rho_0, \rho_1 \supseteq \rho$ such that $f^{-1}(\rho_0) \neq f^{-1}(\rho_1)$ and such that $T(\rho_0)$ and $T(\rho_1)$ are on T^* (these exist by our assumption on T^*) and set $S^*(\sigma * r) = S(f^{-1}(\rho_r))$, $r = 0, 1$. (For definiteness we can preserve lexicographic ordering as well. Uniformity guarantees that this definition is independent of the choice of ρ_0 and ρ_1.) It is clear that $F^* = F \upharpoonright [T^*]$ maps $[T^*]$ onto $[S^*]$ and is induced by some appropriate f^*. (With the above notation if $T^*(\eta_r) = T(\rho_r)$ then $f^*(n) = \operatorname{lth} \eta_i - 1$.)

DEFINITION 1.4. We can now define the notion of forcing $\mathcal{P}$ appropriate to $\mathcal{L}$. A *condition* P consists of a finite $L_P \subseteq \mathcal{L}$ with $0 \in L_P$, trees $T_{P,i}$ for $0 < i \in L_P$ and projection maps $F_{P,j,i}: [T_{P,j}] \to [T_{P,i}]$ induced by functions $f_{P,i,j}$ as above for each $i, j \in L_P$ with $i < j$ such that the maps form a commutative system, i.e.,

$f_{P,i,k} = f_{P,j,k} \circ f_{P,i,j}$ and so $F_{P,k,i} = F_{P,j,i} \circ F_{P,k,j}$ for $i < j < k$ in L_P. Note that for notational convenience we include T_0 which is not a true tree but simply the one branch $T_0(\sigma) = 0^{\text{lth}\,\sigma}$. Similarly the maps $f_{0,i}$ are trivial, i.e., empty, $f_{0,i}^{-1}(\sigma) = 0^{\text{lth}\,\sigma}$ for i the $<$-least element of $L_P - \{0\}$; the other $f_{0,j}^{-1}$ are defined by composition and of course $F_{i,0}(G) = \emptyset$ for every i, G. We say that Q *refines* $P, Q \leqq P$, if $L_Q \supseteq L_P$, $T_{Q,i} \subseteq T_{P,i}$ for $i \in L_P$ and $F_{Q,j,i} = F_{P,j,i} \upharpoonright [T_{Q,j}]$ for $i < j$ in L_P. P and Q are *compatible* if they have a common refinement.

DEFINITION 1.5. If $P \in \mathcal{P}$ we adapt our general definitions of projections (1.3) in the obvious way. Thus for $i < j$ in L_P we say that σ and τ are *congruent* mod(i,j), $\sigma \equiv_{i,j} \tau$, if $\sigma \equiv_{f_{P,i,j}} \tau$ (of course $\sigma \equiv_{0,i} \tau$ for every σ, τ and i) and level n of T_j is *i-differentiating* if it is $f_{P,i,j}$-differentiating. More generally level n of T_k is *(i,j)-differentiating* where $i \leqslant j \leqslant k$ if for σ of length n $\sigma * 0 \equiv_{i,k} \sigma * 1$ but $\sigma * 0 \not\equiv_{j,k} \sigma * 1$. We say that level n of T_k is simply a *j-level* if it is (i,j) differentiating for i the immediate $<$ predecessor of j in L_P.

DEFINITION 1.6. Using the projection trees of Definition 1.3(c) we can define a $Q \leqq P$ with $L_Q = L_p$ by specifying a $T^* \subseteq T_{P,k} = T$ for k the $<$-greatest element of L_p as $T_{Q,k}$ and then simply setting $T_{Q,i} = F_{P,k,i}(T^*)$ for $i < k$ in L_P. We must, of course, begin with a T^* which has infinitely many i-levels for every $0 < i < k$ in L_P. (Level n of T^* is an i-level if for any σ of length n with $T^*(\sigma) = T(\tau)$ and $T^*(\sigma * j) = T(\tau * \rho_j)$, $\tau * \rho_0 \equiv_{i',k} \tau * \rho_1$ but $\tau * \rho_0 \not\equiv_{i,k} \tau * \rho_1$ for i' the $<$-immediate predecessor of i in L_P.)

We can now begin to list the dense sets in $\mathcal{C}$ so that any $\mathcal{C}$-generic filter gives our embedding. We begin with the ones that define G_i.

DEFINITION 1.7. *Totality*: $\mathcal{C}_0$ consists of the sets

$$D_{0,n} = \{P \mid \text{lth}\ T_{P,i}(\phi) \geqq n \text{ for each } i \in L_P\}, \qquad n \in \omega.$$

LEMMA 1.8. *Each $D_{0,n}$ is dense.*

PROOF. Choose any σ such that lth $f_{P,i,k}^{-1}(\sigma) \geqq n$ for i and k the $<$-least and $<$-greatest elements of $L_P - \{0\}$ respectively. We define a $Q \leqq P$ with $L_Q = L_P$ and $Q \in D_{0,n}$ by defining $T_{Q,k} \subseteq T_{P,k}$ and taking projections as in 1.6 above. We just set $T_{Q,k} = \text{Ext}(T_{P,k}, \sigma)$ where

DEFINITION 1.9. Ext(T, σ) is the tree T^* given by $T^*(\tau) = T(\sigma * \tau)$. □

DEFINITION 1.10. *Extendibility*: $\mathcal{C}_1$ contains $\mathcal{C}_0$ and the sets $D_{1,j} = \{P \mid j \in L_P\}$ for $j \in \mathcal{L}$.

LEMMA 1.11. *Each $D_{l,i}$ is dense.*

PROOF. Let $P \in \mathcal{P}$ and $j \notin L_P$ be given. We will define a $Q \leqq P$ where $L_Q = L_P \cup \{j\}$ and $T_{Q,i} = T_{P,i}$, $f_{Q,i,k} = f_{P,i,k}$ and $F_{Q,k,i} = F_{P,k,i}$ for $i < k$ in L_P and $T_{Q,j} =$ identity map on $\mathcal{S}_2$. Thus to completely specify Q it suffices to define the required maps $f_{Q,i,k}$, $i, k \in L_Q$. Let l be the $<$-largest element of L_P. If $l < j$ then we can simply define $f_{Q,l,j}(x) = 2x$. All other maps $f_{Q,i,j}$ are just given by composition: $f_{Q,i,j} = f_{Q,l,j} \circ f_{P,i,l}$. Otherwise let k be the $<$-immediate successor of j in L_Q. We can define $f_{Q,j,l}$ as any monotonic recursive map f such that $\{n \mid \text{level } n \text{ of } T_{P,l} \text{ is an } i\text{-level for } i < k\} \subseteq \operatorname{rg} f \subseteq \{n \mid \text{level } n \text{ of } T_{P,l} \text{ is an } i\text{-level for } i \leqslant k\}$ and such that $\operatorname{rg} f$ is coinfinite in the latter set. All other maps are determined by the commutativity requirements:

$$f_{Q,i,j} = f^{-1} \circ f_{Q,i,l} \qquad \text{for } i < j,$$

$$f_{Q,j,i} = f_{Q,j,l}^{-1} \circ f \qquad \text{for } j < i$$

(where we are using f^{-1} in the usual sense as a partial map from ω to ω).

As these maps are clearly recursive monotonic and have ranges coinfinite where required Q is a forcing condition refining P. □

Note now that if $\mathcal{G}$ is $\mathcal{C}_1$-generic then we can naturally define G_i for $i \in \mathcal{L}$ as $\bigcup\{T_{P,i}(\emptyset) \mid P \in \mathcal{G} \ \& \ i \in L_P\}$ and be assured that G_i is total for every $i \in \mathcal{L}$. Moreover, if $i < j$ then $G_i \leqq_T G_j$ via the $F_{P,i,j}$ specified by any $P \in \mathcal{G}$ with $i, j \in L_P$. (In fact, it is clear that $G_i \leqq_{tt} G_j$.) It is our intention to specify additional dense sets to give a $\mathcal{C} \supseteq \mathcal{C}_1$ such that for any $\mathcal{C}$-generic $\mathcal{G}$ the map sending $i \mapsto \deg(G_i)$ gives an order isomorphism of $\mathcal{L}$ onto an initial segment of $\mathcal{D}$. To facilitate the descriptions of these dense sets we first define forcing.

DEFINITION 1.12. *Forcing.* For any $P \in \mathcal{P}$ and any sentence ϕ of arithmetic with finitely many set parameters G_i, $i \in L_P$, we say that *P forces ϕ*, $P \Vdash \phi$, if for any G on $T_{P,l}$, l the $<$-largest element of L_P, $\phi(G_{i_1}, \dots, G_{i_n})$ holds of the sets $F_{P,l,i_1}(G), \dots, F_{P,l,i_n}(G)$. [Or equivalently in this setting if, for any $\mathcal{C}_1$-generic $\mathcal{G}$ containing P, ϕ is true for the appropriate G_i associated with $\mathcal{G}$.]

Now for the various dense sets required.

DEFINITION 1.13. *Diagonalization.* $\mathcal{C}_2$ contains $\mathcal{C}_1$ and the sets $D_{2,e,i,j} = \{Q \mid j \not\leqslant i \rightarrow Q \Vdash \neg(\phi_e^{G_i} = G_j)\}$ for $e \in \omega$, $i, j \in \mathcal{L}$.

LEMMA 1.14. *The sets $D_{2,e,i,j}$ are dense. Indeed if $i, j \in L_P$ we can find a $Q \leqq P$ with $L_Q = L_P$ and $Q \in D_{2,e,i,j}$.*

PROOF. Let $P \in \mathcal{P}$, $e \in \omega$, and $j \not\leq i$ be given. By Lemma 1.11 we may as well assume that $i, j \in L_P$. Let l be the $<$-largest element of L_P and σ a string on a j-level of $T_{P,l}$. Thus $\sigma * 0 \not\equiv_{j,l} \sigma * 1$ but $\sigma * 0 \equiv_{i,l} \sigma * 1$. Suppose then that

$$T_{P,j}(f^{-1}_{P,j,l}(\sigma *))(x) \neq T_{P,j}(f^{-1}_{P,i,l}(\sigma * 1))(x).$$

If there is no τ on $T_{P,i}$ extending $T_{P,i}(f^{-1}_{P,i,l}(\sigma * 0))$ $(= T_{P,i}(f^{-1}_{P,i,l}(\sigma * 1)) = T_{P,i}(f^{-1}_{P,i,l}(\sigma))$ by choice of σ) for which $\phi^{\tau}_e(x)\downarrow$ then the condition $Q \leqq P$ specified by setting $L_Q = L_P$ and $T_{Q,l} = \operatorname{Ext}(T_{P,l}, \sigma)$ forces $\phi^{G_i}_e(x)\uparrow$ and so is as required. Otherwise let $\tau = T_{P,i}(\rho)$ be be such a string. Choose $k \in \{0, 1\}$ such that

$$\phi^{\tau}_e(x) \neq T_{P,j}(f^{-1}_{P,j,l}(\sigma * k))(x)$$

and $\eta \supseteq \sigma * k$ such that $f^{-1}_{P,i,l}(\eta) = \rho$. If we now let $Q \leqq P$ be determined by setting $L_{Q,l} = \operatorname{Ext}(T_{P,l}, \eta)$ we see that $Q \Vdash \tau \subseteq G_i$ and so $Q \Vdash \phi^{G_i}_e(x) = \phi^{\tau}_e(x)$ while we also have $Q \Vdash T_{P,j}(f^{-1}_{P,j,l}(\sigma * k)) \subseteq G_j$ and so $Q \Vdash \phi^{G_i}_e(x)\downarrow \neq G_j(x)$. □

DEFINITION 1.15. *Initial segments.* $\mathcal{C}_3$ contains $\mathcal{C}_2$ and for $e \in \omega$, $i \in \mathcal{L}$ the sets

$$D_{3,e,i} = \{Q \mid \text{for some } j \leqslant i,\ Q \Vdash \phi^{G_i}_e \text{ is not total or } \phi^{G_i}_e \equiv_T G_j\}.$$

LEMMA 1.16. *The $D_{3,e,i}$ are dense. Indeed if $i \in L_P$ we can find a $Q \leqq P$ with $L_Q = L_P$ and $Q \in D_{3,e,i}$.*

PROOF. Let $P \in \mathcal{P}$, $e \in \omega$, $i \in \mathcal{L}$ be given. We may, of course, assume that $i \in L_P$ and let l be the $<$-largest element of L_P. Moreover we may assume that for every σ and every x there is a $\tau \supseteq \sigma$ such that the condition P' specified by refining the top tree $T_{P,l}$ of P to $\operatorname{Ext}(T_{P,l}, \tau)$ forces $\phi^{G_i}_e(x)\downarrow$. (Otherwise the condition Q specified by refining $T_{P,l}$ to $\operatorname{Ext}(T_{P,l}, \sigma)$ forces $\phi^{G_i}_e(x)\uparrow$ as required.) We now need a definition.

DEFINITION 1.17. $\langle \sigma, \tau \rangle$ gives an e-splitting (of ρ) on T [for S] if $\rho \subseteq \sigma, \tau$ and there is an x such that $\phi^{T(\sigma)}_e(x)\downarrow \neq \phi^{T(\tau)}_e(x)\downarrow$ [and $\sigma \equiv_f \tau$ where f induces a given map $F : [T] \to [S]$]. We call the pair $\langle T(\sigma), T(\tau) \rangle$ an e-splitting (of $T(\rho)$) on T [for S].

SUBLEMMA 1.18. *Suppose now that for some ρ no pair $\langle \sigma, \tau \rangle$ gives an e-splitting of ρ on $T_{P,i}$ for $T_{P,j}$ ($j < i$, $j \in L_P$) and $Q \leqq P$ is given by refining $T_{P,l}$ to* $\operatorname{Ext}(T_{P,l}, \eta)$ *where $f^{-1}_{P,i,l}(\eta) = \rho$ then $Q \Vdash \phi^{G_i}_e \leqq G_j$ or $\phi^{G_i}_e$ is not total.*

PROOF. Let $\mathcal{G}$ be any $\mathcal{C}_1$-generic filter containing Q. Suppose $\phi^{G_i}_e$ is total. Thus for each x there are σ and τ such that $\sigma = T_{Q,i}(\tau) \subseteq G_i$ (and so

$T_{Q,j}(f^{-1}_{Q,j,i}(\tau)) \subseteq G_j)$ and $\phi^{\tau}_e(x)\downarrow$. As there are no e-splittings on $T_{Q,i}$ for $T_{Q,j}$ any τ' for which $\phi_e^{T_{Q,i}(\tau')}(x)\downarrow$ and $T_{Q,j}(f^{-1}_{Q,j,i}(\tau')) \subseteq G_j$ gives the same answer as $\phi^{\tau}_e(x) = \phi^{G_i}_e(x)$. (Otherwise we could extend the shorter of τ, τ' (say τ) to τ'' of the same length as the larger by copying over a final segment (of τ'). The pair $\langle \tau', \tau'' \rangle$ would then give us an e-splitting of ρ on $T_{P,i}$ for $T_{P,j}$ contrary to our assumption.) Thus we can compute $\phi^{G_i}_e(x)$ by simply finding any such τ' — a process clearly recursive in G_j. □

Now let j be the $<$-least element of L_P such that for some ρ there are no e-splittings of ρ on $T_{P,i}$ for $T_{P,j}$. Let $P' \leqq P$ be given by refining $T_{P,l}$ to $\mathrm{Ext}(T_{P,l}, \sigma)$ for a σ with $f^{-1}_{P,i,l}(\sigma) = \rho$. Thus $P' \Vdash \phi^{G_i}_e \leqq_T G_j$ or $\exists x \phi^{G_i}_e(x)\uparrow$. We will now define a $Q \leqq P'$ with $L_Q = L_{P'} = L_P$ such that $Q \Vdash \phi^{G_i}_e \equiv_T G_j$ or $\exists x \phi^{G_i}_x(x)\uparrow$. The idea is to make $T_{Q,i}$ an *e-splitting tree for j*, i.e., $\forall \sigma, \tau$ ($\langle \sigma, \tau \rangle$ give an e-splitting on $T_{Q,i} \Leftrightarrow \sigma \not\equiv_{j,i} \tau$). We have, of course, already insured that if $\sigma \equiv_{j,i} \tau$ then they do not give an e-splitting on $T_{P',i}$ and so not on $T_{Q,i}$ either. We can then use $\phi^{G_i}_e$ to determine the path taken by G_i modulo j, i.e., its projection on $T_{Q,j}$ and so G_j.

To specify Q it suffices to appropriately define a $T^* \subseteq T_{P',l} = T$ with infinitely many k levels for every $0 < k \in L_P$. We define T^* inductively level by level. Suppose $T^*(\sigma)$ is defined for σ of length n. Let $\sigma_0, \sigma_1, \dots, \sigma_{2^n-1}$ list the strings of length n and suppose that we have $\{\tau_s \mid s < 2^n\}$ such that $T^*(\sigma_s) = T(\tau_s)$ with the τ's all of length m. Suppose we now need a k-level in T^*. Consider first the case $j < k$. Let $m_1 + m$ be the next k-level of T and set $T^*(\sigma_s * r) = T(\tau_s * 0^{m_1} * r)$ for $r = 0, 1$. Next suppose $k \leqslant j$. We will define for $r = 0,1$ increasing strings $\rho_{r,\langle s,t\rangle}$ for $s, t < 2^{n-1}$ such that $\langle f^{-1}(\tau_s * \rho_{0,\langle s,t\rangle}), f^{-1}(\tau_t * \rho_{1,\langle s,t\rangle})\rangle$ where $f = f_{P,i,l}$ gives an e-splitting on $T_{P',i}$ for $T_{P',k'}$ where k' is the $<$-immediate predecessor of k. Thus if $T^*(\sigma_s * r) \supseteq T(\tau_s * \rho_{r,2^{n+1}-1})$ we will have all the required splittings. We begin with $\rho_{r,-1} = \emptyset$. Suppose we have defined $\rho_{r,\langle s',t'\rangle} = \mu_r$ and the next step is $\langle s, t\rangle$. We first find η_0, η_1 such that $\langle f^{-1}(\tau_s * \mu_0 * \eta_0), f^{-1}(\tau_s * \mu_0 * \eta_1)\rangle$ gives an e-splitting on $T_{P',i}$ for $T_{P',k'}$ with witness x. We then find an η_2 such that $\mathrm{Ext}(T_{P',l}, \tau_t * \mu_1 * \eta_2)$ forces $\phi^{G_i}_e(x)$ to have some particular value. For definiteness say it differs from that forced by $\tau_s * \mu_0 * \eta_0$. We then set $\rho_{0,\langle s,t\rangle} = \mu_0 * \eta_0$ and $\rho_{1,\langle s,t\rangle} = \mu_1 * \eta_2$. We now have $\rho_{r,2^{n+1}-1} = \nu_r$ for $r = 0, 1$. Let m_r be minimal such that $m_0 + \mathrm{lth}\, \nu_0 = m_1 + \mathrm{lth}\, \nu_1$ is a k level of $T_{P',l}$. We set

$$T^*(\sigma_s * r) = T(\tau_s * \nu_r * 0^{m_r} * r).$$

This completes the construction of the splitting subtree T^* of T and specifies $Q \leqq P'$ by requiring that $T_{Q,l} = T^*$.

Suppose now that $\mathscr{G}$ is $\mathscr{C}_1$-generic, $Q \in \mathscr{G}$ and $\phi_e^{G_i}$ is total. We must show that $G_j \leqq_T \phi_e^{G_i}$. Assume inductively that we have found the ρ of level n such that $T_{Q,j}(\rho) \subseteq G_j$. To decide which of $T_{Q,j}(\rho * 0)$, $T_{Q,j}(\rho * 1) \subseteq G_j$ go to the first k-level, m, in $T^* = T_{Q,I}$ after length $f_{Q,i,I}(\rho)$ for a $k \preccurlyeq j$. Let $\{\sigma_s \mid s < 2^m\}$ list all the elements σ of length m with $f_{Q,j,I}^{-1}(\sigma) = \rho$. Let $g = f_{Q,i,I}^{-1}$. For each $s, s' < 2^m$ $\langle g^{-1}(\sigma_s * 0), g^{-1}(\sigma_{s'} * 1)\rangle$ gives an e-splitting on $T_{Q,i}$. Only one of the answers can agree with $\phi_e^{G_i}$ and so one may be discarded as a possible beginning of G_i. By going through all such pairs we can eliminate either all the $\sigma_s * 0$ or all the $\sigma_s * 1$ as possible beginnings of G_i. Whichever r of 0 and 1 is not so eliminated gives as our next step $g^{-1}(\sigma_s * r) = \rho * r \subseteq G_J$. □

This proof actually shows that for every P, $e \in \omega$ and $i \in L_P$ there is a $Q \leqq P$ (with $L_Q = L_P$) such that there is some x such that $Q \Vdash \phi_e^{G_i}(x)\uparrow$ or $T_{Q,i}$ is an e-splitting tree for some $j \preccurlyeq i$. Given any e we can find a k such that for every A, $\phi_k^A(x)\downarrow$ iff $\phi_e^A(y)\downarrow$ $\forall y \leqq x$ and in this case $\phi_k^A(x) = A(x)$. Applying the above refinement procedure to any P, $i \in L_P$ for k produces a Q such that for some x, $Q \Vdash \phi_e^{G_i}(x)\uparrow$ or $T_{Q,i}$ is an k-splitting tree for i. In the latter case it is clear that $Q \Vdash (\phi_k^{G_i}(x)\downarrow$ for infinitely many $x)$ and so $Q \Vdash \phi_e^{G_i}$ is total.

LEMMA 1.19. *Totality of reducibilities. For $e \in \omega$, $i \in \mathscr{L}$ the sets $D_{4,e,i} = \{Q \mid Q \Vdash (\phi_e^{G_i}$ is total$)$ or for some x $Q \Vdash \phi_e^{G_i}(x)\uparrow\}$ are dense. In fact if $i \in L_P$, $\exists Q \in D_{4,e,i}$ with $Q \leqq P$ and $L_Q = L_P$.* □

PROPOSITION 1.20. *tt-Reducibility. Let $\mathscr{C}_4 \supseteq \mathscr{C}_3$ and all the $D_{4,e,i}$. If $\mathscr{G}$ is $\mathscr{C}_4$-generic and $A \leqq_T G_i$ (for any i) then $A \leqq_{tt} G_i$.*

PROOF. Say $A = \phi_e^{G_i}$. Let $Q \in \mathscr{G} \cap D_{4,e,i}$ so $Q \Vdash \phi_e^{G_i}$ is total. As G_i is on $T_{Q,i}$ and ϕ_e^G is total for every G on $T_{Q,i}$ (as all such are G_i for some $\mathscr{C}_1$-generic $\mathscr{G}$) we can find a k such that $\phi_k^{G_i} = \phi_e^{G_i}$ and ϕ_k^G is total for every G: To compute $\phi_k^G(x)$ compute $\phi_e^G(x)$ and look for an initial segment of G not compatible with $T_{Q,i}$. If the former converges first give its answer as output. If the latter, output 0. □

The point of this proposition is that it guarantees that our embeddings will simultaneously be ones onto initial segments of the *wtt* and *tt*-degrees.

THEOREM 1.21. *If $\mathscr{G}$ is $\mathscr{C}_4$-generic then the map $i \mapsto \deg_r(G_i)$ is an order isomorphism onto an initial segment of the r-degrees for $r = T$, wtt or tt.*

PROOF. $\mathscr{C}_1$-genericity guarantees that if $i \preccurlyeq j$ then $G_i \leqq_{tt} G_j$. $\mathscr{C}_2$-genericity guarantees that if $i \not\preccurlyeq j$ then $G_i \not\leqq_T G_j$. $\mathscr{C}_3$-genericity guarantees that if $G \leqq_T G_i$ then $G \equiv_T G_j$ for some $j \preccurlyeq i$ while $\mathscr{C}_4$-genericity guarantees that if $G \leqq_T G_i$ then

$G \leqq_{tt} G_i$. Thus the G_i give an initial segment for any degree relation between tt and T. $\square$

Our goal now is to extend this result to linear orderings $\mathcal{L}^*$ of size $\aleph_1$ with least element 0 and the countable predecessor property. We begin by dividing $\mathcal{L}^*$ up as $\bigcup_{\alpha<\omega_1} \mathcal{L}_\alpha$ where $\{\mathcal{L}_\alpha\}$ is a monotonic continuous sequence of countable downward closed suborderings of $\mathcal{L}^*$ with no last element. Our plan is to define a class $\mathscr{C}_5 \supseteq \mathscr{C}_4$ of dense sets and a sequence of forcing notions $\mathscr{P}_\alpha$, each contained in the one generated for $\mathcal{L}_\alpha$ above, and corresponding $\mathscr{C}_5$-generic filters $\mathscr{G}_\alpha \subseteq \mathscr{P}_\alpha$ such that the $\mathscr{G}_\alpha$ form a continuous monotonic sequence. Given any such sequence $\{\mathscr{G}_\alpha\}$ we can then define the map $i \mapsto \deg(G_{\alpha,i})$ for any α with $i \in \mathcal{L}_\alpha$. This map, of course, then gives an isomorphism of $\mathcal{L}^*$ onto an initial segment of the degrees.

The idea is to put into $\mathscr{P}_{\alpha+1}$ only those conditions associated with $\mathcal{L}_{\alpha+1}$ which are already appropriately represented in $\mathscr{G}_\alpha$. To define the method of representing a $P \in \mathscr{P}_{\alpha+1}$ by a $P' \in \mathscr{P}_\alpha$ we first need some notation.

DEFINITION 1.22. Let $P \in \mathscr{P}$ be a notion of forcing appropriate to some $\mathcal{L}$ and let ϕ be an $<$-preserving partial 1–1 map which maps L_P onto $L \subseteq \mathcal{L}$, with $\phi(0) = 0$. $\phi(P)$ is the $Q \in \mathscr{P}$ with $L_Q = L$, $T_{Q,i} = T_{P,\phi^{-1}(i)}$, $F_{Q,j,i} = F_{P,\phi^{-1}(j),\phi^{-1}(i)}$ and $f_{Q,i,j} = f_{P,\phi^{-1}(i),\phi^{-1}(j)}$ for $i, j \in L_Q$. In particular, we can restrict a condition P to a smaller ordering $L \subseteq L_P$ in the obvious way by setting $P \restriction L = \phi(P)$ where $\operatorname{dom} \phi = L$ and $\phi \restriction L = \mathrm{id} \restriction L$. Thus, for example, for every P and $L \subseteq L_P$ $P \leqq P \restriction L$ and so generic filters are closed under restrictions.

We can now define our $\mathscr{P}_\alpha$, $\mathscr{G}_\alpha$ by induction. Let $\mathscr{P}_0$ be the notion of forcing defined above for $\mathcal{L}_0$. Suppose $\mathscr{P}_\alpha$ is defined. Let $\mathscr{G}_\alpha$ be a $\mathscr{C}_5$-generic filter for $\mathscr{P}_\alpha$ (we will verify later that one such exists by induction). Now let $\mathscr{P}_{\alpha+1}$ be all those conditions P in the notion of forcing for $\mathcal{L}_{\alpha+1}$ for which there is a $P' \in \mathscr{G}_\alpha$ and a one-one partial map ϕ such that range $\phi = L_P$, $\phi \restriction \mathcal{L}_\alpha \cap L_P = \mathrm{id}$ and $\phi(P') = P$. Of course for a limit ordinal λ we set $\mathscr{P}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{P}_\alpha$ and $\mathscr{G}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{G}_\alpha$.

The crucial step now is to define the class of dense sets needed to make $\mathscr{C}_5$-genericity of $\mathscr{G}_\alpha$ imply the existence of a $\mathscr{C}_5$-generic $\mathscr{G}_{\alpha+1} \subseteq \mathscr{P}_{\alpha+1}$. The density of the $D_{0,n}$ (totality), $D_{2,i,j}$ (diagonalization) and $D_{4,e,i}$ (totality of reductions) in $\mathscr{P}_{\alpha+1}$ follow immediately from the corresponding genericity requirements on $\mathscr{G}_\alpha$. Problems arise only for the $D_{1,i}$ and $D_{3,e,i}$.

Consider first an $R \in \mathscr{P}_{\alpha+1}$ with witnesses R' and ϕ as in the definition of $\mathscr{P}_{\alpha+1}$. As $R' \restriction \operatorname{dom} \phi \in \mathscr{G}_\alpha$ we may as well assume that $L_{R'} = \operatorname{dom} \phi$. Let $\{i_1, \dots, i_s\} = L_R \cap \mathcal{L}_\alpha = \phi^{-1}(L_R \cap \mathcal{L}_\alpha) \subseteq L_R$ and let $\{j_1, \dots, j_n\} = \phi^{-1}(L_R - \mathcal{L}_\alpha)$ be the rest of $L_{R'}$. Given an $i \in \mathcal{L}_{\alpha+1} - L_R$ we wish to find a $Q \leqq R$, $Q \in \mathscr{P}_{\alpha+1}$, with $i \in L_Q$. If $i \preccurlyeq j_1$ then as $j_1 \in \mathscr{L}_\alpha$ and $\mathscr{L}_\alpha$ is downward closed, $i \in \mathscr{L}_\alpha$ and so there is no problem. We can simply choose any $Q' \leqq R'$, $Q' \in \mathscr{G}_\alpha$ with $i \in L_{Q'}$. We can then define ψ by $\psi(x) = x$ for $x \in L_{Q'}$, $x \preccurlyeq i$ and $\psi(j_t) = \phi(j_t)$ for $t \leqq n$. $Q = \psi(Q') \in \mathscr{P}_{\alpha+1}$ by definition while $i \in L_Q$ and $Q \leqq R$ as required. If, however, $j_1 \preccurlyeq i$ there are problems.

Suppose first that $i \in \mathscr{L}_\alpha$. We can, of course, find a $Q' \leqq R'$ with $Q' \in \mathscr{G}_\alpha$ and $i \in L_{Q'}$. We cannot, however, extend ϕ to ψ by setting $\psi(i) = i$ to get $\psi(Q') = Q$ as ψ would then not preserve order or not be one-one. Thus we must also add on to $L_{Q'}$ new elements $k_1, \dots, k_n$ all $> i$ to represent the elements of $L_R - \mathscr{L}_\alpha$. We could then hope to set $\psi(x) = x$ for $x \preccurlyeq i$, $x \in L_{Q'}$ and $\psi(k_t) = \phi(j_t)$ for $t \leqq n$ to get an element Q of $\mathscr{P}_{\alpha+1}$ with $i \in L_Q$. The requirement that $Q \leqq R$ thus becomes one that $\theta(Q') \leqq R'$ where $\theta(k_t) = j_t$, $t \leqq n$ and $\theta(i_t) = i_t$ for $t \leqq s$.

Now if $i \in \mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha$ then we must insert an additional k into the list $k_1, \dots, k_n$ at the appropriate, say mth, place. To do this it suffices that there be room for such an insertion since we can then just apply the extendibility property of $\mathscr{G}_\alpha$. All these considerations lead to the definition of the $D_{5,L,m,i,R}$ below. The point is that if $\mathscr{G}_\alpha$'s genericity requirements include the $D_{5,L,m,i,R}$, then the $D_{1,i}$ will be dense in $\mathscr{P}_{\alpha+1}$.

Next suppose (with R, R', ϕ as above) that we are given an e and wish to find a $Q \leqq R$, $Q \in \mathscr{P}_{\alpha+1}$ which for some $i \in L_Q$, $i \preccurlyeq \phi(j_m)$ forces [$\phi e^{G_{\phi(j_m)}}$ is of the same degree as G_i or is not total]. We cannot simply take any $Q' \leqq R'$ which for some $i \in L_{Q'}$, $i \preccurlyeq j_m$ forces [$\phi_e^{G_{j_m}}$ is of the same degree as G_i or is not total] since that i may not be in dom ϕ. Indeed it may not be possible to extend ϕ to include i in its domain (e.g. $j_1 < i < j_2$ but $\nexists x(\phi(j_1) < x < \phi(j_2))$). Thus we must find a $Q' \in \mathscr{G}_\alpha$ with possibly new elements $k_1, \dots, k_n \in L_{Q'}$ to represent $L_R - \mathscr{L}_\alpha$ such that for some $i \in L_{Q'}$, $i < k_m$, $Q' \Vdash \phi_e^{G_{k_m}} \equiv G_i$ or is not total. The crucial point, however, is that we must be able to define a θ on all of $L_{Q'}$ with $\theta(k_t) = \phi(j_t)$ to give us a condition $Q = \theta(Q') \leqq R$ such that $Q \in \mathscr{P}_{\alpha+1}$ and $Q \Vdash \phi_e^{G_{\phi(j_m)}} \equiv G_{\theta(i)}$ or is not total. If the $\{k_1, \dots, k_n\}$ form a final segment of $L_{Q'}$ then we can define θ by $\theta(k_t) = \phi(j_t)$ and $\theta(x) = x$ for $x \in L_{Q'}$, $x < k_1$. The requirement that $Q = \theta(Q') \leqq R$ then becomes that $\psi(Q') \leqq R'$ where $\psi(k_t) = j_t$, $t \leqq n$ and $\psi(i_t) = i_t$, $t \leqq s$.

These considerations lead to the definition of the $D_{5,L,m,R,e}$ below. Again if the genericity requirements of $\mathscr{G}_\alpha$ force it to meet each $D_{5,L,m,R,e}$ we will be able to prove that the $D_{3,e,i}$ are dense in $\mathscr{P}_{\alpha+1}$.

We revert now to our original notation so that $\mathscr{P}$ is the notion of forcing associated with a countable ordering $\mathscr{L}$.

DEFINITION 1.23. *Amalgamation.* $\mathscr{C}_5$ consists of $\mathscr{C}_4$ plus for each $i \in \mathscr{L}$, each finite $L \subseteq \mathscr{L}$, $L = \{j_1 < \cdots < j_n\}$, each $m \leqq n$ and each $R \in \mathscr{P}$ with L a final segment of L_R the sets

$$D_{5,L,m,i,R} = \{Q \mid Q \text{ is incompatible with } R \text{ or } [Q \leqq R \ \& \ (\exists k_0 < k_1 < \cdots < k_m < k < k_{m+1} < \cdots < k_{n+1} \text{ in } L_Q) (j_n,\ i < k_0 \text{ and if we define } \phi(k_s) = j_s \text{ for } 1 \leqq s \leqq n \text{ and } \phi(l) = l \text{ for } l \in L_{R-L} \text{ then } \phi(Q) \leqq R)\}$$

and for each $e \in \omega$ the sets

$$D_{5,L,m,R,e} = \{Q \mid Q \text{ is incompatible with } R \text{ or } [Q \leqq R \ \& \ (\exists k_1 < k_2 < \cdots < k_n \text{ forming a final segment of } L_Q) [j_n < k_1 \text{ and if we define } \phi(k_s) = j_s \text{ for } 1 \leqq s \leqq n \text{ and } \phi(i) = i \text{ for } i \in L_R - L \text{ then } \phi(Q) \leqq R \ \& \text{ for some } i \in L_Q,\ i \preccurlyeq k_m\ Q \Vdash \phi_e^{G_{k_m}} \text{ is not total or } \phi_e^{G_{k_m}} \equiv_T G_i]\}.$$

The combinatorial fact needed to prove that these sets are dense is given by the following:

LEMMA 1.24. *For any $P \in \mathscr{P}$ with $\{i_1 < \cdots < i_s\} = L$ a final segment of $L_P = \{j_1, \ldots, j_n\} \cup L$ and any $k_1 < \cdots < k_s$ with $i_s < k_1$ there is a $Q \leqq P$ with $\{k_1, \ldots, k_s\} \subseteq L_Q$ such that $\phi(Q) \leqq P$ where $\phi(k_t) = i_t$ for $t \leqq s$ and $\phi \upharpoonright L_P - L =$* id.

PROOF. To refine P (without regard to extending L_p) just means to give a subtree of T_{P,i_s} which has j and i-differentiating levels for each $j, i \in L_p$ as the trees for the other elements of L_p and the associated maps are then all determined by the projections associated with P. If in addition we wish to extend L_p to $L_Q = L_p \cup \{k_1, \ldots, k_s\}$ we must define T_{Q,k_s} and the maps giving T_{Q,k_t}, $t < s$, and the relations to the $T_{Q,i}$. If we are to have $\phi(Q) \leqq P$ as well, then T_{Q,k_s} must be a subtree of T_{p,i_s} and the T_{Q,k_t} (and associated maps from T_{Q,k_s}) must be given by the maps from T_{p,i_s} to T_{p,i_t}. Thus to specify Q it suffices to properly define $T'' = T_{Q,i_s}$ and $T' = T_{Q,k_s}$ (each subtree of $T = T_{P,i_s}$) and $f_{Q,i_s,k_s} = f$ as the rest of the condition will be determined by the existing maps and commutativity requirements. A key point here is that f_{Q,i_s,k_1} is to be determined by composing f with the projection from $T' = T_{Q,k_s}$ to T_{Q,k_1} by the map f^{-1}_{P,i_s,i_1}. Thus levels in T' dedicated to i or j differentiations must involve splits which in T are not congruent mod i_1.

We begin by setting $T'(\emptyset) = T''(\emptyset) = T(\emptyset)$. Suppose we have defined T' and f^{-1} up through level n, lth $\sigma = n$, $T'(\sigma) = T(\alpha)$, $f^{-1}(\sigma) = \tau$, $T''(\tau) = T(\beta)$ and

$g^{-1}(\alpha) = g^{-1}(\beta) * 0^y$ for some y where $g = f_{P,j_n,i_s}$. We define the next level of T' by cases:

(i) We need a j-level for $j \in L_P - L$. Let $m_1 > \text{lth}\,\alpha$, $\text{lth}\,\beta$ be least such that m_1 is a j-level of T. Now set, for $r = 0, 1$,

$$T'(\sigma * r) = T(\alpha * 0^{(m_1 - \text{lth}\,\alpha)} * r),$$

$$T''(\tau * r) = T(\beta * 0^{(m_1 - \text{lth}\,\beta)} * r) \qquad \text{and}$$

$$f^{-1}(\sigma * r) = \tau * r \qquad (\text{so } f(\text{lth}\,\tau) = \text{lth}\,\sigma).$$

It is clear that $\text{lth}\,\sigma$ is a j-level in T' and

$$g^{-1}(\alpha * 0^{(m_1 - \text{lth}\,\alpha)} * r) = g^{-1}(\beta * 0^{m_1 - \text{lth}\,\beta} * r).$$

(ii) We need an i-level for $i \in L$. Let $m_1 > \text{lth}\,\alpha$, $\text{lth}\,\beta$ be least such that it is an i-level in T and let $m_2 > m_1$ be least such that it is an i_1-level in T. Now set

$$T'(\sigma * r) = T(\alpha * 0^{m_2 - \text{lth}\,\alpha} * r),$$

$$T''(\tau * r) = T(\beta * 0^{m_1 - \text{lth}\,\beta} * r * 0^{m_2 - m_1}) \qquad \text{and}$$

$$f^{-1}(\sigma * r) = \tau * r \qquad (\text{so } f(\text{lth}\,\tau) = \text{lth}\,\sigma).$$

Again it is clear that $\text{lth}\,\sigma$ is an i-level in T' and that

$$g^{-1}(\alpha * 0^{m_2 - \text{lth}\,\alpha} * r) = g^{-1}(\beta * 0^{m_1 - \text{lth}\,\beta} * r * 0^{m_2 - m_1}).$$

(It is here that we see the effects of having to work within the $\preccurlyeq i_s$-differentiating levels of T to get ones that are $\preccurlyeq i_s$-differentiating in T'.)

(iii) We need a k_t-level for $t \leqq s$. Let $m_1 > \text{lth}\,\alpha$, $\text{lth}\,\beta$ be the next i_t-level in T. Set $T'(\sigma * r) = T(\alpha * 0^{m_1 - \text{lth}\,\alpha} * r)$ and $f^{-1}(\sigma * r) = \tau$ (so $\text{lth}\,\sigma \notin \text{rg}\,f$). Of course $\text{lth}\,\sigma$ is now a k_t-level in T'. The twist in this case is that $g^{-1}(\alpha * 0^{m_1 - \text{lth}\,\alpha} * r) = g^{-1}(\beta) * 0^y$ where y is the number of elements between $\text{lth}\,\beta$ and m_1 in the range of g.

We now define a condition Q by setting $L_Q = L_P \cup \{k_1, \ldots, k_s\}$, $T_{Q,k_s} = T'$, $T_{Q,i_s} = T''$, $f_{Q,i_s,k_s} = f$ and all other trees and maps are given by the projections determined in P and commutativity requirements. As $T_{Q,i_s} \subseteq T_{P,i_s}$ and the rest of $Q \restriction L_P$ is defined by the projections in P it is clear that $Q \leqq P$.

We next claim that if $\phi(i) = i$ for $i \in L_P - L$ and $\phi(k_m) = i_m$ for $m \leqq s$ then $\phi(Q) \leqq P$. The definitions clearly show that $T_{\phi(Q),l} \subseteq T_{P,l}$ for $l \in L_P = L_{\phi(Q)}$ and that the maps between trees within $L_P - L$ or L are the restrictions of those in P. Thus we need only check the maps between an element in $L_P - L$ and one in L.

By commutativity it suffices to check that $F_{\phi(Q),i_s,j_n} = F_{Q,k_s,j_n} = F_{P,i_s,j_n} \upharpoonright T_{Q,k_s}$. Now by definition

$$F_{Q,k_s,j_n} = F_{Q,i_s,j_n} \circ F_{Q,k_s,i_s} = F_{P,i_s,j_n} \circ F_{Q,k_s,i_s}.$$

But this is guaranteed to be $F_{P,i_s,j_n} \upharpoonright T_{Q,k_s}$ by the part of our construction that says that at infinitely many levels n (all except k_t ones) we have for lth $\sigma = n$, τ, α and β such that $T_{Q,k_s}(\sigma) = T_{P,i_s}(\alpha)$, $T_{Q,i_s}(\tau) = T_{P,i_s}(\beta)$, $f^{-1}_{Q,i_s,k_s}(\sigma) = \tau$ and $f^{-1}_{P,j_n,i_s}(\alpha) = f^{-1}_{P,j_n,i_s}(\beta)$. The point here is that

$$f^{-1}_{Q,j_n,i_s} \circ f^{-1}_{Q,i_s,k_s}(\sigma) = f^{-1}_{Q,j_n,i_s}(\tau) = f^{-1}_{P,j_n,i_s}(\beta) = f^{-1}_{P,j_n,i_s}(\alpha)$$

which gives $F_{P,i_s,j_n} \upharpoonright T_{Q,k_s}$. □

LEMMA 1.25. *The $D_{5,L,m,i,R}$ are each dense.*

PROOF. Let L_0, m_0, i_0, R_0 be given as in the definition of $D_{5,L,m,i,R}$ and consider any $P' \in \mathcal{P}$. If P' is incompatible with R_0, $P' \in D_{5,L_0,m_0,i_0,R_0}$. Otherwise let P be a common extension. Let $L = \{i_1 < \cdots < i_s\}$ consist of all elements of L_P which are $\geqslant$ any of L_0. Now choose $k_0 < \cdots < k_{s+1}$ with $k_0 \geqslant i_0$, max L and such that $\exists k$, s_0, $s_1(k_{s_0} \leqslant k \leqslant k_{s_1}$ and $i_{s_0} = j_{m_0})$ and apply Lemma 1.24 to get a $Q \leqq P$ with $\phi(Q) \leqq P$ where $\forall t \leqq s(\phi(k_t) = i_t)$ & $\phi \upharpoonright L_P - L = \mathrm{id}$. By the proof of Lemma 1.11 we can get a $Q' \leqq Q$ with $L_{Q'} = L_Q \cup \{k_0, k, k_{s+1}\}$ such that $Q' \big| L_Q = Q$. Thus $Q' \leqq P \leqq P'$, $\phi(Q') \leqq P \leqq P'$ and Q' is our desired extension of P' in D_{5,L_0,m_0,i_0,R_0}. □

LEMMA 1.26. *The $D_{5,L,m,R,e}$ are each dense.*

PROOF. Fix L_0, m_0, R_0 and e_0 as in the definition of $D_{5,L,m,R,e}$ and consider any $P' \in \mathcal{P}$. Again we need only consider the case where we have a common extension P of P' and R_0. Let $L = \{i_1 < \cdots < i_s\}$, $k_1 < \cdots < k_s$ and Q be as in the proof of Lemma 1.25. Now let $Q' = Q \upharpoonright (L_{R_0} \cup L_{P'} \cup \{k_{s_1}, k_{s_2}, \ldots, k_{s_n}\})$ where $i_{s_l} = j_l$. Thus $Q' \leqq P'$, R_0 and $\phi(Q') \leqq R_0$ where $\phi(k_{s_l}) = j_l$ and $\phi \upharpoonright L_{R_0} - L_0 = \mathrm{id}$. Let $s_{m_0} = j$, $e_0 = c$ and apply Lemma 1.16 to get a $Q'' \leqq Q'$ with $L_{Q''} = L_{Q'}$ such that for some $i \in L_{Q''}$, $Q'' \Vdash \phi_c^{G_{k_j}} \equiv_T G_i$ or $\phi_c^{G_{k_j}}$ is not total. As $L_{Q''} = L_{Q'}$, $\{k_{s_1}, \ldots, k_{s_n}\}$ is a final segment of $L_{Q''}$. Moreover, as $Q'' \leqq Q'$, $\phi(Q'') \leqq \phi(Q') \leqq R_0$ as well. □

We now know that there are $\mathscr{C}_5$-generic $\mathscr{G}_0$ for $\mathcal{P}_0$. The next step is an induction on α. We have already motivated the proofs of density for the $D_{0,n}$, $D_{1,i}$, $D_{2,i,j}$, $D_{3,e,i}$ and $D_{4,e,i}$ in $\mathcal{P}_{\alpha+1}$ based on the $\mathscr{C}_5$-genericity of $\mathscr{G}_\alpha$. The new idea needed is that although the additional requirements in $\mathscr{C}_5$ were designed to prove these facts they also suffice to propagate themselves. The proof that the

$D_{5,L,m,i,R}$ are dense in $\mathcal{P}_{\alpha+1}$ is basically a straightforward application of the same genericity requirement in $\mathcal{G}_\alpha$ to the representative P' (and ϕ) in $\mathcal{G}_\alpha$ of a condition $P \in \mathcal{P}_{\alpha+1}$. For the $D_{5,L,m,R,e}$ one really needs pictures. Roughly speaking, however, one first applies one instance of these requirements on level α to move the representative $\phi^{-1}(L)$ in P' out to the end (making no use of the initial segment requirement). One then applies another instance of these requirements (this time we need the initial segment restriction as well) to produce yet a further refinement which contains a copy of the representatives of $L_P - \mathcal{L}_\alpha$ in P' followed by these new elements as a final segment of the resulting condition. This condition in $\mathcal{G}_\alpha$ then represents the required $Q \leqq P$.

LEMMA 1.27. *If $\mathcal{G}_\alpha$ is $\mathcal{C}_5$-generic for $\mathcal{P}_\alpha$ then there is a $\mathcal{G}_{\alpha+1}$ which is $\mathcal{C}_5$-generic for $\mathcal{P}_{\alpha+1}$.*

PROOF. We must show that each of the required sets is dense in $\mathcal{P}_{\alpha+1}$. Consider $P \in \mathcal{P}_{\alpha+1}$. Let $P' \in \mathcal{P}_\alpha$ and ϕ be as in the definition of $\mathcal{P}_{\alpha+1}$.

(a) $D_{0,n}$. Let $Q' \leqq P'$ be given by $\mathcal{C}_0$-genericity of $\mathcal{G}_\alpha$, i.e., $Q' \in \mathcal{G}_\alpha \cap D_{0,n}$. $\phi(Q') \leqq \phi(P') = P$ and $\phi(Q')$ is clearly in $D_{0,n}$. Thus $\phi(Q')$ is the desired element of $\mathcal{P}_{\alpha+1} \cap D_{0,n}$ extending P. □

(b) $D_{1,j}$. Let $L = \{j_1 < \cdots < j_n\}$ be the final segment of P' containing $\phi^{-1}[\mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha]$.

First suppose $j \in \mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha$. Let m be such that $\phi(j_m) < j < \phi(j_{m+1})$ and choose $Q' \leqq P'$ with $Q' \in \mathcal{G}_\alpha \cap D_{5,L,m,i,P'}$ $(i = j_n)$. Thus if ϕ' is as in the definition of $D_{5,L,m,i,P'}$, $\phi'(Q') \leqq P'$. Thus $\phi\phi'(Q') \leqq \phi(P') = P$. We now extend $\phi\phi'$ to ψ by setting $\psi(k) = j$. Thus $\psi(Q') \leqq P$, $j \in L_{\psi(Q')}$ and by definition $\psi(Q') \in \mathcal{P}_{\alpha+1}$.

Next suppose $j \in \mathcal{L}_\alpha$. Choose $Q' \leqq P'$ with $Q' \in \mathcal{G}_\alpha \cap D_{5,L,0,j,P'}$ with ϕ' as in the definition of $D_{5,L,0,j,P'}$ so that $\phi'(Q') \leqq P'$. Thus $\phi\phi'(Q') \leqq \phi(P') = P$. Now choose $Q'' \leqq Q'$ with $Q'' \in \mathcal{G}_\alpha \cap D_{1,j}$. Thus $\phi\phi'(Q'') \leqq P$. Extend $\phi\phi'$ to ψ by setting $\psi(j) = j$, so $\psi(Q'') \leqq P$, $j \in L_{\psi(Q'')}$ and $\psi(Q'') \in \mathcal{P}_{\alpha+1}$. □

(c) $D_{2,e,i,j}$ for $j \not\leq i$. We may assume by (b) that $i, j \in L_P$. Choose $Q' \leqq P'$ with $Q' \in \mathcal{G}_\alpha \cap D_{2,e,\phi^{-1}(i),\phi^{-1}(j)}$. $\phi(Q') \in \mathcal{P}_{\alpha+1}$, $\phi(Q') \leqq \phi(P') = P$ and clearly $\phi(Q') \Vdash \neg(\phi_e^{G_i} = G_j)$. In fact,

NOTE 1.28. If $\phi(P') = P \in \mathcal{P}_{\alpha+1}$ and Ψ is a sentence mentioning only G_i for $i \in \operatorname{dom} \phi$ and $P' \Vdash \Psi$ then by the definition of forcing $\phi(P') \Vdash \phi(\Psi)$ where $\phi(\Psi)$ is gotten by replacing each G_i by $G_{\phi(i)}$. □

(d) $D_{3,e,i}$. We may assume by (b) that $i \in L_P$. Let $\{j_1 < \cdots < j_n\} = L$ be the final segment of P' containing $\phi^{-1}[\mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha]$. Choose $Q' \leqq P'$ with $Q' \in \mathcal{G}_\alpha \cap D_{5,L,m,P',e}$ where $j_m = \phi^{-1}(i)$. Again we let ϕ', $\{k_1 < \cdots < k_n\}$ and i'

witness that $Q' \in D_{5,L,m,P',e}$ so that $\phi\phi'(Q') \leqq P$. We now extend $\phi\phi'$ to a ψ defined on all of $L_{Q'}$ by setting $\psi(j) = j$ for $j < k_1$, $j \in L_{Q'}$. Thus $\psi(Q') \in \mathcal{P}_{\alpha+1}$ and $\psi(Q') \leqq \phi\phi'(Q') \leqq P$. Finally as $Q' \Vdash \phi_e^{G_{k_m}}$ is not total or $\phi_e^{G_{k_m}} \equiv_T G_{i'}$, $\psi(Q') \Vdash \phi_e^{G_i}$ is not total or $\phi_e^{G_i} \equiv G_{\psi(i')}$. □

(e) $D_{4,e,i}$. Choose any $Q' \leqq P'$ with $Q' \in D_{4,e,\phi^{-1}(i)} \cap \mathcal{G}_\alpha$. $\phi(Q') \leqq \phi(P') = P$, $Q = \phi(Q') \in \mathcal{P}_{\alpha+1}$ and as $Q' \Vdash \phi_e^{G_{\phi^{-1}(i)}}$ is total or, for some x, $Q' \Vdash \phi_e^{G_{\phi^{-1}(i)}}(x)\uparrow$, $Q \Vdash \phi_e^{G_i}(x)$ is total or, for some x, $G \Vdash G_e^{G_i}(x)\uparrow$ as required. □

(f) $D_{5,L,m,i,R}$. We need only consider the case that R and P have a common refinement $S \in \mathcal{P}_{\alpha+1}$. Let S', ϕ show that $S \in \mathcal{P}_{\alpha+1}$. Let $L' = \{j'_1, \ldots, j'_{n^*}\}$ be the final segment of $L_{S'}$ beginning with $\phi^{-1}(j_1) = j'_1$. Let m' be such that $\phi(j'_{m'}) = j_m$. Let $i' = i$ if $i \in \mathcal{L}_\alpha$ and otherwise set $i' = j'_{n^*}$. Now choose a $Q' \leqq S'$, $Q' \in \mathcal{G}_\alpha \cap D_{5,L',m',i',S'}$ and let $\phi', k'_1, \ldots, k'_{n^*}, k'$ be the appropriate witnesses. Thus $\phi'(Q') \leqq S'$ so $\phi\phi'(Q') \leqq S \leqq R, P$. Now choose any appropriately ordered $k_1, \ldots, k_{n^*}, k$ with $i, j_n < k_1$ and extend ϕ to ψ by setting $\psi(k'_s) = k_s$. Now $Q' \leqq S'$ and so $\psi(Q') \leqq \psi(S') = \phi(S') = S \leqq R$ and $\psi(Q') \in \mathcal{P}_{\alpha+1}$. Of course, $j_n, i \leqslant k_1$. Moreover if we define θ by $\theta \upharpoonright L_R - L = \mathrm{id}$ and $\theta(k_{t(s)}) = j_s$ for $t(s)$ such that $\phi(j'_{t(s)}) = j_s$ then $\theta, k_{t(1)}, \ldots, k_{t(n)}$ and k witness that $\psi(Q') \in D_{5,L,m,i,R}$. The only point left to verify is that $\theta(\psi(Q')) \leqq R$. Now $(\theta\psi(Q')) \upharpoonright L_R = (\phi\phi'(Q')) \upharpoonright L_R$ and so as $\phi\phi'(Q') \leqq R$, $\theta(\psi(Q')) \leqq R$. (Verification: If $i \in L_R - L$ then $(\theta\psi)^{-1}(i) = \psi^{-1}\theta^{-1}(i) = \psi^{-1}(i) = \phi^{-1}(i)$ while $(\phi\phi')^{-1}(i) = (\phi')^{-1}\phi^{-1}(i) = \phi^{-1}(i)$ since ϕ' is the identity on $S' - L' \supseteq \phi^{-1}(L_R - L)$. If $i \in L$ then $i = j_s$ for some s and $\psi^{-1}\theta^{-1}(j_s) = \psi^{-1}(k_{t(s)}) = k'_{t(s)}$ while $(\phi')^{-1}\phi^{-1}(j_s) = (\phi')^{-1}(j'_{t(s)}) = k'_{t(s)}$.)

□

(g) $D_{5,L,m,R,e}$. Again we let $S \leqq R$, $P, S' \in \mathcal{G}_\alpha$ and ϕ witness $S \in \mathcal{P}_{\alpha+1}$ and $L' = \{\phi^{-1}(j_1), \ldots, \phi^{-1}(j_n)\} = \{j'_1, \ldots, j'_n\} \subseteq L_{S'}$. Note that at the cost of replacing S' with $S' \upharpoonright \mathrm{dom}\,\phi$ we can assume that $L_{S'} = \mathrm{dom}\,\phi$. Now choose $Q' \leqq S'$ with $Q' \in \mathcal{G}_\alpha \cap D_{5,L',m,R',e}$ where $L_{R'} = \phi^{-1}(L_R)$, $L' = \phi^{-1}(L) = \{j_1, \ldots, j'_n\}$ and $R' = S' \upharpoonright L_{R'}$ (so $\phi(R') \leqq R$). Let ϕ' and $k'_1, \ldots, k'_n$ be the required witnesses: $\phi'(k'_s) = j'_s = \phi^{-1}(j_s)$, $\phi'(i) = i$ for $i \in L_{R'} - L'$ and $\phi'(Q') \leqq R'$. Next let

$$L'' = \phi^{-1}(\mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha) \cup \{k'_1, \ldots, k'_n\}, \; L_{R''} = L_{S'} \cup \{k'_1, \ldots, k'_n\} \text{ and } R'' = Q' \upharpoonright L_{R''}$$

(so $\phi'(Q' \upharpoonright L_{R''}) \leqq R'$). Next choose $Q'' \leqq Q'$ with $Q'' \in \mathcal{G}_\alpha \cap D_{5,L'',m'',R'',e}$ where k'_m is the m''th element of L''. Let $\{i'_1, \ldots, i'_t\} = \phi^{-1}(\mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha)$, $\{i_1, \ldots, i_t\} = L_S \cap (\mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha)$ and let ϕ'', $i''_1, \ldots, i''_t, k''_1, \ldots, k''_n$ and $i_0 \in L_{Q''}$ be the witnesses for $Q'' : \phi''(i''_s) = i'_s$, $\phi''(k''_s) = k'_s$, $\phi''(i) = i$ for $i \in L_{R''} - L'' = L_{S'} - \phi^{-1}(\mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha)$ and $\phi''(Q'') \leqq R''$. We can now define a $P'' \in \mathcal{P}_{\alpha+1}$ with a witness ψ such that $\psi(Q'') = P''$ by setting $\psi(i) = i$ for $i < i''_1$, $\psi(i''_s) = i_s$ and

$\psi(k_s'') = k_s$ where we can choose any $k_1, \dots, k_s$ in $\mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha$ above all elements of L_S. We claim that $P'' \leqq S \leqq P$ and that $P'' \in D_{5,L,m,R,e}$ as required:

(i) $P'' \leqq S \leqq P$: We know that $\phi''(Q'') \leqq R'' \leqq S'$ and so $\phi\phi''(Q'') \leqq \phi(S') = S$. Thus it suffices to prove that $\phi\phi'' = \psi$ on $\psi^{-1}(L_S)$. If $i \in L_S$, $i \in \mathscr{L}_\alpha$ (i.e., $i < i_1'$) then all of ψ, ϕ and ϕ'' are the identity on i. Consider then some $i_s \in L_S \cap (\mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha)$. $\psi^{-1}(i_s) = i_s''$ but $\phi\phi''(i_s'') = \phi(i_s') = i_s$ as well.

(ii) As $Q'' \Vdash (\phi_e^{G_{k_m''}} \equiv_T G_{i_0}$ or is not total), $\psi(Q'') = P'' \Vdash (\phi_e^{G_{k_m}} \equiv G_{\psi(i_0)}$ or is not total).

(iii) Let $\psi'(k_s) = j_s$, $\psi'(i) = i$ for $i \in L_R - L$. We must show that $\psi'(P'') \leqq R$ to finish the verification.

The preimages of $\psi'(P'') \upharpoonright L_R$ in Q'' are given by $j_s \mapsto k_s''$ for $j_s \in L$, $i \mapsto i$ for $i \in (L_R - L) \cap \mathscr{L}_\alpha$ and $i_s \mapsto i_s''$ for $i_s \in (L_R - L) - \mathscr{L}_\alpha$. We claim that $\phi\phi'\phi''$ is the inverse of this map so that $\phi\phi'\phi''(Q'') \upharpoonright L_R = \psi'(P'') \upharpoonright L_R$:

(1) $\phi\phi'\phi''(k_s'') = \phi\phi'(k_s') = \phi(j_s') = j_s$ for $j_s \in L$.

(2) $\phi\phi'\phi''(i) = \phi\phi'(i)$ for $i \in L_{R''} - L'' = L_{S'} - \phi^{-1}(\mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha)$
$= \phi(i)$ if $i \in L_{R'} - L' = \phi^{-1}(R) - \phi^{-1}(L)$ as well
$= i$ if $i \in \mathscr{L}_\alpha$

thus $\phi\phi'\phi''(i) = i$ if all of these conditions hold: $i \in L_{S'} \cap L_{R'} \cap \mathscr{L}_\alpha - \phi^{-1}(\mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha) - \phi^{-1}(L)$ but $L_{R'} \subseteq L_{S'}$ and $L_{R'} = \phi^{-1}(L_R) \subseteq \mathscr{L}_\alpha$ so we need $i \in L_{R'} - \phi^{-1}(\mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha) - \phi^{-1}(L)$. Now $\phi = \mathrm{id}$ on $L_R \cap \mathscr{L}_\alpha$ so

$$L_{R'} - \phi^{-1}(\mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha) = L_R \cap \mathscr{L}_\alpha.$$

Thus we need $i \in L_R \cap \mathscr{L}_\alpha - \phi^{-1}(L)$ but again on $\mathscr{L}_\alpha$, $\phi^{-1} = \mathrm{id}$ and so this is the same as $i \in (L_R - L) \cap \mathscr{L}_\alpha$.

(3) $\phi\phi'\phi''(i_s'') = \phi\phi'(i_s')$ for all $s \leqq t$
$= \phi(i_s')$ for $i_s' \in L_{R'} - L'$
$= i_s$ for $i_s \in L_S - \mathscr{L}_\alpha$

so $\phi\phi'\phi''(i_s'') = i_s$ if $i_s \in L_R - L - \mathscr{L}_\alpha$.

Finally we have $\phi''(Q'') \leqq R'' = Q' \upharpoonright L_{R''}$ and so $\phi'\phi''(Q'') \leqq \phi'(Q' \upharpoonright L_{R''}) \leqq R'$ and at last $\phi\phi'\phi''(Q'') \leqq \phi(R') = R$. □

THEOREM 1.29. *If $\mathscr{L}^*$ is a linear ordering (with least element) of size $\aleph_1$ with the countable predecessor property, then there are G_i for $i \in \mathscr{L}^*$ such that the G_i give initial segments of the T, wtt and tt degrees isomorphic to $\mathscr{L}^*$.*

PROOF. Define $\mathscr{L}_\alpha$, $\mathscr{P}_\alpha$ and $\mathscr{G}_\alpha$ as described above. (Note that $\mathscr{G}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{G}_\alpha \subseteq \mathscr{P}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{P}_\alpha$ is $\mathscr{C}_5$-generic by the monotonicity of the sequence and the $\mathscr{C}_5$-genericity of each $\mathscr{G}_\alpha$, $\alpha < \lambda$.) The G_i are then given also as described above ($G_i = \bigcup_{P \in \mathscr{G}} T_{P,i}(\varnothing)$, $\bigcup_{\alpha<\omega_1} \mathscr{G}_\alpha = \mathscr{G}$). As in the countable case the $\mathscr{C}_4$-

genericity of each $\mathcal{G}_\alpha$ guarantees that the G_i, $i \in \mathcal{L}_\alpha$, give an initial segment of each type of degree isomorphic to $\mathcal{L}_\alpha$. Thus their union gives one isomorphic to $\mathcal{L}^*$. □

2. Countable upper semi-lattices

Our goal now is to prove that every u.s.l. with 0 of size $\aleph_1$ satisfying the countable predecessor property (c.p.p.) is isomorphic to an initial segment of the Turing (*tt* and *wtt*) degrees. In outline we will follow the path laid down for the case of linear orderings in Section 1. This section gives a presentation of the countable case designed for our extension process. Except for not having fixed a greatest element in our approximations and so having trees for each element of the (u.s.) lattice rather than one master tree we essentially follow Lerman [10]. Other than rearranging some of the definitions the only difference comes in expanding conditions to add on new elements (density of the $D_{1,i}$) and the related requirements on the representations of $\mathcal{L}$.

The major difference between the case of countable linear orderings (or distributive lattices) and arbitrary (countable) lattices or upper semi-lattices appears in the coding scheme used to guarantee that if $i < j$ then $G_i \leqq_T G_j$. In Section 1 (and similarly for distributive lattices as in Lachlan [7]) the ordering of $L_P \subseteq \mathcal{L}$ is represented by inclusion on a class of sets (the ranges of the functions $f_{P,i,l}$ where l is the $<$-largest element of L_P). Of course one cannot represent a non-distributive lattice in this way. [Another view of this problem is presented in Lachlan [7]. The reductions of G_i to G_j for $i < j$ given in Section 1 are in fact $m-1$ reductions. Thus if $i < j$ then $G_i \leqq_m G_j$ and so the map $i \mapsto \deg_m(G_i)$ gives an embedding into the m-degrees (actually to an initial segment of the m-degrees). There are, however, no such embeddings of non-distributive lattices.] Thus we must use some more complicated [at least *tt*] coding to reflect the ordering of $\mathcal{L}$ in the general case.

We use Lerman's u.s.l. tables:

DEFINITION 2.1. *U.S.L. tables.* Let $\mathcal{L}$ be a finite u.s.l. with 0 (and hence a lattice with 1).

(a) $\Theta \subseteq \omega^{\mathcal{L}}$ is an *(u.s.l) table for* $\mathcal{L}$ iff

(i) $\forall \alpha, \beta \in \Theta \ (\alpha(0) = \beta(0))$,

(ii) $\forall \alpha, \beta \in \Theta \ \forall x, y \in \mathcal{L}[x \leqslant y \ \& \ \alpha(y) = \beta(y) \rightarrow \alpha(x) = \beta(x)]$,

(iii) $\forall \alpha, \beta \in \Theta \ \forall x, y, z \in \mathcal{L}[x \vee y = z \ \& \ \alpha(x) = \beta(x) \ \&$
$\alpha(y) = \beta(y) \rightarrow \alpha(z) = \beta(z)]$,

(iv) $\forall x, y \in \mathcal{L}[x \not\leqslant y \rightarrow \exists \alpha, \beta \in \Theta \ (\alpha(y) = \beta(y) \ \& \ \alpha(x) \neq \beta(x))]$.

If Θ satisfies (i)–(iii) but not necessarily (iv) we call it a *positive u.s.l. table* for $\mathscr{L}$.

(b) If $\mathscr{L}' \subseteq \mathscr{L}$ and Θ is a table for $\mathscr{L}$ then $\Theta \restriction \mathscr{L}'$ is the obvious table for $\mathscr{L}'$ (i.e. $\{\alpha \restriction \mathscr{L}' \mid \alpha \in \Theta\}$. If $x \in \mathscr{L}$ we write $\Theta \restriction x$ for $\{\alpha(x) \mid \alpha \in \Theta\}$.

(c) Without loss of generality we may assume that $\alpha(0) = 0$ for every $\alpha \in \Theta$.

(d) If $\alpha, \beta \in \Theta$ and $x \in \mathscr{L}$ we say that α is congruent to β modulo x, $\alpha \equiv_x \beta$, iff $\alpha(x) = \beta(x)$.

Note that every finite u.s.l. $\mathscr{L}$ with 0 has a finite (u.s.l.) table (Lerman [10, Appendix B.2.2]). Our plan is to use trees with branchings given by a table Θ for $\mathscr{L}$ so that the listed requirements will guarantee that (i) $G_0 \equiv_T \emptyset$; (ii) $x \leqslant y \to G_x \leqq_T G_y$; (iii) $x \vee y = z \to G_x \oplus G_y \leqq_T G_z$; and (iv) allow for the possibility that $x \not\leqslant y \to G_x \not\leqq_T G_y$. Before we can define the required trees, however, we must first handle infimum requirements and then allow for the need to extend the finite lattices in a condition within the table itself.

DEFINITION 2.2. *Sequential tables.*

(a) If Θ and Ψ are tables for $\mathscr{L}$ then Ψ *extends* Θ if $\Theta \subseteq \Psi$ and Ψ is an *admissible extension* of $\Theta, \Theta \subseteq_a \Psi$, if in addition

$$\forall \alpha \in \Psi \; \exists \beta \in \Theta \; \forall \gamma \in \Theta \; \forall x \in \mathscr{L} \; [\alpha \equiv_x \gamma \to \alpha \equiv_x \beta].$$

(Note that this relation is transitive.)

(b) $\Theta = \{\Theta_i \mid i < \omega\}$ is a *sequential* (*weakly homogeneous*) *table* for $\mathscr{L}$ iff

(i) $\forall i \in \omega$ (Θ_i is a finite table for $\mathscr{L}$).

(ii) $\forall i \in \omega$ ($\Theta_i \subseteq_a \Theta_{i+1}$).

(iii) $\forall i \in \omega \; \forall \alpha, \beta \in \Theta_i \; \forall x, y, z \in \mathscr{L} \; [x \wedge y = z \;\&\; \alpha \equiv_z \beta \to \exists \gamma_0, \gamma_1, \gamma_2 \in \Theta_{i+1} \; (\alpha \equiv_x \gamma_0 \equiv_y \gamma_1 \equiv_x \gamma_2 \equiv_y \beta)]$.

(iv) $\forall i \in \omega \; \forall \alpha_0, \alpha_1, \beta_0, \beta_3 \in \Theta_i \; [\forall x \in \mathscr{L}(\alpha_0 \equiv_x \alpha_1 \to \beta_0 \equiv_x \beta_3) \to \exists \beta_1, \; \beta_2 \in \Theta_{i+1} \; \exists f_0, f_1, f_2 : \Theta_i \to \Theta_{i+1} \; (f_0(\alpha_0) = \beta_0 \;\&\; f_0(\alpha_1) = \beta_1 \;\&\; f_1(\alpha_0) = \beta_1 \;\&\; f_1(\alpha_1) = \beta_2 \;\&\; f_2(\alpha_0) = \beta_2 \;\&\; f_2(\alpha_1) = \beta_3 \;\&\; \forall y \in \mathscr{L} \; \forall \alpha, \beta \in \Theta_i \; (\alpha \equiv_x \beta \to f_0(\alpha) \equiv_x f_0(\beta) \;\&\; f_1(\alpha) \equiv_x f_1(\beta) \;\&\; f_2(\alpha) \equiv_x f_2(\beta)))]$.

Condition (iii) is designed to handle $\wedge$ requirements and (iv), the weak homogeneity property, plays a more technical role connected to initial segment requirements that need not concern us. We should point out, however, that (iv) is taken from Lerman [9, p. 268] rather than Lerman [10, p. 278] or Lachlan and Lebeuf [8, p. 289] since one actually needs three functions rather than two.

DEFINITION 2.3. *Extendible tables.*

(a) If Θ and Ψ are tables for $\mathscr{L}$ then $p = \{p_x \mid x \in \mathscr{L}\}$ is an *isomorphism* of Θ onto Ψ, $p : \Theta \xrightarrow{\sim} \Psi$, if each p_x is a recursive one-one function with recursive range such that $\{p(\alpha) \mid \alpha \in \Theta\} = \Psi$ where by definition $(p(\alpha))(x) = p_x(\alpha(x))$ for every $x \in \mathcal{L}$. If there is such a p we say that Θ and Ψ are *isomorphic*, $\Theta \simeq \Psi$, and write $p[\Theta] = \Psi$. (Note that isomorphisms preserve congruence relations (i.e., $\alpha \equiv_x \beta \Leftrightarrow p(\alpha) \equiv_x p(\beta)$.)

(b) A sequential table $\{\Theta_i\}$ for $\mathcal{L}$ is *extendible* if it satifies the following conditions:

(v) For any finite $\mathcal{L}' \supseteq \mathcal{L}$ and any table Ψ for $\mathcal{L}'$ there is a $j \in \omega$, a table Ψ^* for $\mathcal{L}'$ and a $p : \Psi \xrightarrow{\sim} \Psi^*$ so that the following diagram is correct:

$$\begin{array}{l} \Psi \\ {\scriptstyle p}\big\downarrow{\scriptstyle \wr} \\ \Psi^* \to \Psi^* \restriction \mathcal{L} \subseteq_a \Theta_j \end{array}$$

(vi) For every $i < j \in \omega$, every finite $\mathcal{L}' \supseteq \mathcal{L}$, every table Ψ for $\mathcal{L}'$ such that $\Psi \restriction \mathcal{L} \subseteq_a \Theta_i$ and every table Ψ^* for $\mathcal{L}'$ such that $\Psi \subseteq_a \Psi^*$, there is a $k > j$ and a $p : \psi^* \xrightarrow{\sim} \psi^+$ such that $p(\alpha) = \alpha$ for $\alpha \in \Psi$, $\forall x \in \mathcal{L}' \; \forall n [n \notin \Psi \restriction x \to p_x(n) > j]$ and such that the following diagram commutes:

$$\begin{array}{lcl} \Psi \to \Psi \restriction \mathcal{L} & \hookrightarrow_a & \Theta_i \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle a}\big\downarrow \\ \Psi^* & & \Theta_j \\ {\scriptstyle p}\big\downarrow{\scriptstyle \wr} & & {\scriptstyle a}\big\downarrow \\ \Psi^+ \to \Psi^+ \restriction \mathcal{L} & \hookrightarrow_a & \Theta_k \end{array}$$

(c) A sequential table $\Theta = \{\Theta_i \mid i \in \omega\}$ is *recursive* if there is a recursive function giving canonical indices for the Θ_i (as $\mathcal{L}$ is finite we may choose any identification with a subset of ω to formally define recursiveness on the appropriate space).

(d) If Θ is a sequential table for $\mathcal{L}$ (i.e., $\Theta(i) = \Theta_i$) we write $\Theta \restriction \mathcal{L}' = \{\Theta_i \restriction \mathcal{L}' \mid i \in \omega\}$ and $\Theta \restriction x = \{\Theta_i \restriction x \mid i \in \omega\}$ for $\mathcal{L}' \subseteq \mathcal{L}$ and $x \in \mathcal{L}$. (Note that $\Theta \restriction x$ is thus a map $\omega \to [\omega]^{<\omega}$.)

(e) If Θ is a sequential table for $\mathcal{L}$ and Ψ is one for $\mathcal{L}' \supseteq \mathcal{L}$ then Ψ *refines* Θ if there is a recursive h such that $\Psi_i \restriction \mathcal{L} \subset_a \Theta_{h(i)}$.

We can now define the types of trees that will make up our forcing conditions. The idea is that given a sequential table Θ for $\mathcal{L}$ the tree appropriate for an $x \in \mathcal{L}$ is a $\Theta \restriction x$-tree. The projections F between branches are then explicitly given by the tables. If the tree for x follows the path along $\alpha(x)$ then the tree for $y < x$ follows the one along $\alpha(y)$. Of course Definition 2.1(ii) guarantees that this is well defined, i.e., knowing $\alpha(x)$ is sufficient to determine $\alpha(y)$ — one needn't know α (i.e., the path on the tree for $1_{\mathcal{L}}$).

DEFINITION 2.4. *The notion of forcing.* Let $\mathscr{L}$ be a countable u.s.l. with least element 0. We define the *notion of forcing* $\mathscr{P}$ *appropriate to* $\mathscr{L}$ as follows.

(a) A *condition* P consists of a finite sub u.s.l. L_P of $\mathscr{L}$ containing 0; a recursive extendible sequential table $\Theta_P = \{\Theta_{P,i} \mid i \in \omega\}$ for L_P; for each $x \in L_P$ a uniform recursive $\Theta_P \restriction x$-tree and a commutative system of recursive maps $F_{P,x,y} : [T_x] \to [T_y]$ for each $y < x$ in $\mathscr{L}_P$ which are induced by Θ_P in the sense that if $G_x = T_x[g]$ then $F_{P,x,y}[G_x] = G_y$ is $T_y[h]$ where $h(n) = \alpha(y)$ for any $\alpha \in \Theta_n$ such that $\alpha(x) = g(n)$ (this is well defined by Definition 2.1(ii)).

(b) A condition Q *refines* one P, $Q \leqq P$, if $L_Q \supseteq L_P$, $T_{Q,x} \subseteq T_{P,x}$ for $x \in L_P$, $F_{Q,x,y} = F_{P,x,y} \restriction [T_{Q,x}]$ for $y < x$ in L_P.

(c) The *restriction of* P *to* $L \subseteq L_P$, $P \restriction L$, is the condition Q such that $L_Q = L$, $\Theta_Q = \Theta_P \restriction L$, $T_{Q,x} = T_{P,x}$ and $F_{Q,x,y} = F_{P,x,y}$ for $x, y \in L$.

The typical method for specifying a refinement Q of P with $L_P = L_Q = L$ and $\Theta_P = \Theta_Q = \Theta$ is to give an appropriate subtree of $T = T_{P,1}$ (where we use 1 to denote the *greatest element of* $\mathscr{L}_P$) and then take the "projections" as the subtrees of $T_{P,x}$ for $x \in \mathscr{L}$. Recall that in general we may specify a subtree T^* of the (uniform) $\Theta \restriction 1$-tree T by giving a (uniform) $\Theta \restriction 1$-tree S and setting $T^* = T \circ S$. In order for the projections to be well defined and generate a refinement of P, S must satisfy an extra condition.

DEFINITION 2.5. *Subtrees and projections.* Let Θ be a sequential table for $\mathscr{L}$ and T be a uniform $\Theta \restriction 1$-tree.

(a) If $x < y$, z in $\mathscr{L}$ and $\sigma \in \mathscr{S}_{\Theta \restriction y}$, $\tau \in \mathscr{S}_{\Theta \restriction z}$ we say that σ *is congruent to* τ mod x, y, z, $\sigma \equiv_{x,y,z} \tau$, if for each $n < \mathrm{lth}\,\sigma$, $\mathrm{lth}\,\tau$ and each $\alpha, \beta \in \Theta_n$ with $\alpha(y) = \sigma(n)$ and $\beta(z) = \tau(n)$ we have that $\alpha(x) = \beta(x)$, i.e., $\alpha \equiv_x \beta$. If y and z are clear from the context we will frequently write this as $\sigma \equiv_x \tau$.

(b) If $x < y$ in $\mathscr{L}$ and $\sigma \in \mathscr{S}_{\Theta \restriction y}$ then the y*-projection of* σ *on* x, $f_{y,x}(\sigma)$, is that $\tau \in \mathscr{S}_{\Theta \restriction x}$ with the same length as σ such that $\sigma \equiv_x \tau$ (i.e., $\sigma \equiv_{x,y,x} \tau$). Again if y is clear from the context we often omit it and call τ *the projection of* σ *on* x, $f_x(\sigma)$.

(c) A uniform $\Theta \restriction 1$-tree, S, is *distinguished* if

$$\forall x \in \mathscr{L}\ \forall \sigma, \tau \in \mathscr{S}_{\Theta \restriction 1}\ [\sigma \equiv_x \tau \Leftrightarrow S(\sigma) \equiv_x S(\tau)].$$

(d) If Θ, L and T come from a condition P (i.e., $\Theta = \Theta_P$, $L = L_P$ and $T = T_{P,1}$) and S is a distinguished $\Theta \restriction 1$-tree we can define a condition $Q = S(P) \leqq P$ by setting $L_Q = L$, $\Theta_Q = \Theta$, $F_{Q,x,y} = F_{P,x,y} \restriction [T_{Q,x}]$ and $T_{Q,x} = T_{P,x} \circ S_x$ where we define S_x by $S_x(\sigma) = f_x(S(\tau))$ for any $\tau \in \mathscr{S}_{\Theta \restriction 1}$ such that $f_x(\tau) = \sigma$. S_x is well defined since S is distinguished. Similarly, the maps $F_{Q,x,y}$ are induced by $\Theta_Q = \Theta$ as required.

(e) With this notation we can describe the functions $F_{P,x,y}$ by noting that $F_{P,x,y}(T_{P,x}[g]) = T_{P,y}[f_{x,y}g]$.

The simplest example of this type of refinement is given by taking $T \circ S$ to be the extension subtree of T above some $\sigma \in \mathcal{S}_{\Theta\restriction 1}$.

DEFINITION 2.6. *Extension trees.* With notation as above we let $\mathrm{Ext}(T, \sigma)$ for $\sigma \in \mathcal{S}_{\Theta\restriction 1}$ be $T \circ S$ where $S(\tau) = \sigma * \tau$ which is clearly a distinguished uniform $\Theta \restriction 1$-tree.

We can now begin to list the dense sets $\mathscr{C}$ that guarantee that any $\mathscr{C}$-generic filter gives our required embedding.

DEFINITION 2.7. *Totality.* $\mathscr{C}_0$ consists of the sets $D_{0,n} = \{P \mid \mathrm{lth}(T_{P,x}(\phi)) \geqq n$ for each $x \in L_P\}$.

LEMMA 2.8. *Each $D_{0,n}$ is dense.*

PROOF. Let $P \in \mathcal{P}$. Let $Q \leqq P$ be defined as in Definition 2.5(a) by setting $T_{Q,1} = \mathrm{Ext}(T_{P,1}, \sigma)$ for any $\sigma \in \mathcal{S}_{\Theta\restriction 1}$ such that $\mathrm{lth}\, f_x(\sigma) \geqq n$ for every $x \in L_P$. □

LEMMA 2.9. *If Θ is a recursive sequential table for $\mathcal{L}$ and $\mathcal{L}'$ is a finite extension of $\mathcal{L}$ then there is a recursive sequential table Ψ for $\mathcal{L}'$ which refines Θ.*

PROOF. This is a special case of Theorem 4.1 whose statement and proof we defer. □

DEFINITION 2.10. *Extendibility.* $\mathscr{C}_1$ contains $\mathscr{C}_0$ and the sets $D_{1,x} = \{P \mid x \in L_P\}$ for $x \in \mathcal{L}$.

LEMMA 2.11. *Each $D_{1,x}$ is dense.*

PROOF. Consider $P \in \mathcal{P}$ and $x \in \mathcal{L} - L_P$. Let L be the (finite) sub u.s.l. of $\mathcal{L}$ generated by L_P and x. By Lemma 2.9 we can choose Ψ to be a recursive sequential table for L refining Θ_P via the recursive function h. We will define a $Q \leqq P$ with $L_Q = L$ and $\Theta_Q = \Psi$. The trees $T_{Q,y}$ for $y \in L_Q - L_P$ will just be the $\Psi \restriction y$-identity trees. For $x \in L_P$ we define a $\Psi \restriction x$-tree $T_{Q,x} \subseteq T_{P,x} : T_{Q,x}(\emptyset) = T_{P,x}(0^{h(0)})$ and if $T_{Q,x}(\sigma)$ is defined as $T_{P,x}(\tau)$ with $\mathrm{lth}\, \sigma = n$, $\mathrm{lth}\, \tau = h(n)$ and $i \in \Psi_n \restriction x \subseteq \Theta_{h(n)} \restriction x$ then $T_{Q,x}(\sigma * i) = T_{P,x}(\tau * i^{h(n+1)-h(n)})$. It is easy to see from the definition of Ψ refining Θ that the maps $F_{Q,x,y}$ for $y < x$ in L_P induced by Ψ are precisely the restrictions of $F_{P,x,y}$ to $[T_{Q,x}]$. Thus $Q \leqq P$ as required. □

Now note that if $\mathcal{G}$ is $\mathscr{C}_1$-generic we can naturally define functions G_x for each $x \in \mathcal{L}$ as $\bigcup\{T_{P,x}(\emptyset) \mid P \in \mathcal{G}\ \&\ x \in L_P\}$, i.e., $\mathcal{G}_x(n) = T_{P,x}(\emptyset)(n)$ for any

$P \in \mathscr{G} \cap D_{0,n} \cap D_{1,x}$. The G_x are well defined by the compatibility requirement on generic filters and are total for each $x \in \mathscr{L}$ by the density of the $D_{0,n}$ and $D_{1,x}$. Now as in Section 1 if $y < x$ then $G_y \leqq_T G_x$ via any $F_{P,x,y}$ with $P \in \mathscr{G}$, $x, y \in L_P$. Moreover if $x \vee y = z$ then $G_x \oplus G_y \equiv G_z$. Of course $G_x \oplus G_y \leqq_T G_z$ by our first observation. To see that $G_z \leqq_T G_x \oplus G_y$ consider any $P \in \mathscr{G}$ with $x, y, z \in L_P$ so that $G_x \in [T_{P,x}]$, $G_y \in [T_{P,y}]$ and $G_z \in [T_{P,z}]$. Suppose that $G_x = T_{P,x}[g_x]$, $G_y = T_{P,y}[g_y]$ and $G_z = T_{P,z}[g_z]$. Thus $g_x = f_{z,x}g_z$ and $g_y = f_{z,y}g_z$ where the projections are defined by Θ_P. Now by clause (iii) of Definition 2.1, $f_{z,x}g_z$ and $f_{z,y}g_z$ uniquely determine g_z. As the trees are recursive we can therefore calculate G_z from $G_x \oplus G_y$. Thus any $\mathscr{C}_1$-generic $\mathscr{G}$ determines a map $\mathscr{L} \to \mathscr{D}$ given by $x \mapsto \deg(G_x)$ which preserves $\leqq$ and $\vee$. We must now specify additional collections of dense sets which will make this mapping one-one and its range an initial segment of $\mathscr{D}$. We define forcing as before.

Definition 2.12. *Forcing.* For any $P \in \mathscr{P}$ and any sentence $\phi(\underline{G}_{x_1}, \ldots, \underline{G}_{x_n})$ of arithmetic with function parameters $\underline{G}_{x_i}$, $x_i \in L_P$ we say that P *forces* ϕ, $P \Vdash \phi$, if for any G on $T_{P,1}$, $\phi(G_{x_1}, \ldots, G_{x_n})$ is true where $G_{x_i} = F_{P,1,x_i}[G]$.

Definition 2.13. *Diagonalization.* $\mathscr{C}_2$ contains $\mathscr{C}_1$ and for every $e \in \omega$, $x, y \in \mathscr{L}$ the sets $D_{2,e,x,y} = \{Q \mid x \not\leqslant y \to Q \Vdash \neg(\phi_e^{G_x} = G_y)\}$.

Lemma 2.14. *The $D_{2,e,x,y}$ are dense and indeed we can find a $Q \leqq P$ as required with $L_Q = L_P$ and $\Theta_Q = \Theta_P$.*

Proof. This is essentially the same as the proof of Lemma 1.14. Alternatively assume $x, y \in L_P$ and let $T = T_{P,1}$. Lemma VII.2.5 of Lerman [10] gives a $T^* \subseteq T$ via a distinguished tree S (an extension tree) such that the condition Q determined by T^* as in Definition 2.5(d) is as required. □

Definition 2.15. *Initial segments.* $\mathscr{C}_3$ contains $\mathscr{C}_2$ and for each $e \in \omega$, $x \in \mathscr{L}$ the sets $D_{3,e,x} = \{Q \mid \text{for some } y \leqslant x,\ Q \Vdash (\phi_e^{G_x} \text{ is not total or } \phi_e^{G_x} \equiv_T G_y)\}$.

Lemma 2.16. *The $D_{3,e,x}$ are dense. Indeed if $e \in \omega$ and $x \in L_P$, we can find a $Q \leq P$ in $D_{3,e,x}$ with $L_Q = L_P$ and $\Theta_Q = \Theta_P$.*

Proof. Let $P \in \mathscr{P}$, $e \in \omega$ and $x \in L_P$ be given. Let $T = T_{P,1}$. Section 3 of chapter VII of Lerman [10] is entirely devoted to the proof that (with very slight notational changes) there is a $T^* \subseteq T$ (given by a distinguished tree S) such that the $Q \leqq P$ with $L_Q = L_P$, $\Theta_Q = \Theta_P$ specified by setting $T^* = T_{Q,1}$ is as required. □

We now have enough dense sets to embed $\mathscr{L}$ as an initial segment of $\mathscr{D}$.

THEOREM 2.17. *If $\mathcal{G}$ is $\mathcal{C}_3$-generic then the mapping* $x \mapsto \deg(G_x)$ *gives an u.s.l. isomorphism of* $\mathcal{L}$ *onto an initial segment of* $\mathcal{D}$.

PROOF. $\mathcal{C}_1$-genericity guarantees that the map is an u.s.l. homomorphism; $\mathcal{C}_2$-genericity that it is one-one; and $\mathcal{C}_3$-genericity that it is it is onto an initial segment.

REMARK 1.18. As the $T_{P,x}$ for $P \in \mathcal{G}$, $x \in \mathcal{L}$ are finitely branching with the branching given recursively we can recursively code the G_x as sets so as to make it possible to consider *tt* reducibilities as well. One can then easily define $\mathcal{C}_4$ to contain the appropriate sets $D_{4,e,x}$ and prove their density as in Section 1 to get the same result for *tt* and *wtt*-reducibilities. We omit the details and will continue to omit them in the next section.

3. Size $\aleph_1$ upper semi-lattices

We now wish to extend an embedding as in Section 2 to an u.s.l. $\mathcal{L}^*$ of size $\aleph_1$ with 0 and the countable predecessor property. Let us try to follow the procedure used for linear orderings in Section 1. Thus we first divide $\mathcal{L}^*$ up into a monotonic continuous sequence $\{\mathcal{L}_\alpha\}$ of downward closed sub u.s.l.'s so that $\mathcal{L}^* = \bigcup_{\alpha<\aleph_1} \mathcal{L}_\alpha$. We then hope to define a class of dense sets $\mathcal{C}_5$ and a sequence of forcing notions $\mathcal{P}_\alpha$ each contained in the one appropriate for $\mathcal{L}_\alpha$ and a corresponding continuous sequence of generic filters $\mathcal{G}_\alpha \subseteq \mathcal{P}_\alpha$. Again we want $\mathcal{P}_{\alpha+1}$ to contain conditions which are represented in $\mathcal{G}_\alpha$.

DEFINITION 3.1. *Isomorphisms.* Let $P \in \mathcal{P}$, a notion of forcing appropriate to some countable u.s.l. $\mathcal{L}$ with 0, and let ϕ be a partial u.s.l. monomorphism which maps L_P onto some $L \subseteq \mathcal{L}$ with $\phi(0)=0$. $\phi(P)$ is the condition $Q \in \mathcal{P}$ with $L_Q = L$, $T_{Q,x} = T_{P,\phi^{-1}(x)}$, $F_{Q,x,y} = F_{P,\phi^{-1}(x),\phi^{-1}(y)}$ for $x, y \in L$ and $\Theta_Q = \phi(\Theta_P)$ where $\phi(\Theta_P)(n) = \phi(\Theta_P(n)) = \{\phi(\alpha) \mid \alpha \in \Theta_P(n)\}$ and $\phi(\alpha)(x) = \alpha(\phi^{-1}(x))$ for $x \in L$.

There are now, however, a number of difficulties with defining $\mathcal{P}_{\alpha+1}$ as simply those conditions P for which there is a $P' \in \mathcal{G}_\alpha$ and a ϕ with $\operatorname{rg}\phi = L_P$ and $\phi \restriction \mathcal{L}_\alpha \cap L_P = \mathrm{id}$ such that $\phi(P') = P$. The first problem arises in trying to prove the extendibility lemma, i.e., the density of the $D_{1,x}$. There just may not be any $L_{P'} \subseteq \mathcal{L}_0$, say, which is isomorphic to some $L \subseteq \mathcal{L}_1$. Thus we could never hope to get a condition $P \in \mathcal{P}_1$ with $L \subseteq L_P$. The obvious solution is to require that the $\mathcal{L}_\alpha$ be elementary submodels of $\mathcal{L}^*$.

Unfortunately this refinement does not seem to suffice to prove the initial segment lemma — the density of the $D_{3,e,x}$. To understand the difficulty suppose we have a $P \in \mathcal{P}_1$ represented by $P' \in \mathcal{G}_0$ with $\phi(P') = P$. We are given an $x \in L_P - \mathcal{L}_0$ and an $e \in \omega$ and wish to refine P to a Q which forces $\phi_e^{G_x}$, if total, to be of the same degree as G_y for some $y < x$, $y \in L_Q$. To do this we need a $Q' \leqq P'$, $Q' \in \mathcal{G}_0$ and a ϕ_1 with $\phi_1(Q') = Q$ such that Q' forces $\phi_e^{G_{\phi^{-1}(x)}}$, if total, to be of the same degree as $G_{\phi^{-1}(y)}$ for some $y \leqslant x$, $y \in L_Q$. The trouble is that each possible candidate z for $\phi_1^{-1}(x)$ (i.e., those at least bearing the same relationship to the elements of $L_P \cap \mathcal{L}_0$ that x does) could well have elements below it (in $\mathcal{L}_0$) which are not below x. (It is easy enough to arrange such a situation.) Moreover it could also be that any condition $Q' \in \mathcal{G}_0$ which decides the degree of $\phi_e^{G_z}$, for any such z, forces it to be that of some G_y with $y \mid x$. For such a situation there can be no $Q' \in \mathcal{G}_0$ with a ϕ_1 giving $\phi_1(Q') = Q \in \mathcal{P}_1$ as required.

The solution is to represent conditions $P \in \mathcal{P}_1$ only via maps ϕ and conditions $P' \in \mathcal{G}$ such that no extraneous elements in $\mathcal{L}_0$ are below $\phi^{-1}(x)$ for any $x \in L_P - \mathcal{L}_0$. As there may be no such representatives in $\mathcal{L}_0$ we must add them on. One cannot simply put in more and more elements of $\mathcal{L}^*$ since this would make $\mathcal{L}_0$ uncountable. Thus we will extend $\mathcal{L}^*$ via a saturation process that puts in isomorphic copies of all possible finite extensions L' of any finite sub u.s. lattices L which add below the elements of L' only elements generated by joining elements of L' with ones below elements of L. These elements must exist and cannot ruin our representation if the ordering is defined in the natural way (freely). Of course if we expect an embedding of our extension of $\mathcal{L}^*$ as an initial segment of $\mathcal{D}$ to include one of $\mathcal{L}^*$ we must make sure that it is an end extension as well.

We hope that this discussion in some way motivates the following definitions and lemmas.

DEFINITION 3.2. *Special extensions.* Let $\mathcal{L}$ be an u.s.l. and $\mathcal{L}_0$, $\mathcal{L}_1$ each a finite sub u.s.l. of $\mathcal{L}$. We say that *$\mathcal{L}_1$ is a special extension of $\mathcal{L}_0$ in $\mathcal{L}$, $\mathcal{L}_0 \subseteq_{\text{sp}} \mathcal{L}_1(\mathcal{L})$*, if

(i) *$\mathcal{L}_1$ is an end-extension of $\mathcal{L}_0$ in $\mathcal{L}$, $\mathcal{L}_0 \subseteq_{\text{end}} \mathcal{L}_1(\mathcal{L})$*, i.e., $\forall x \in \mathcal{L}_1\ \forall y \in \mathcal{L}_0\ (x < y \rightarrow x \in \mathcal{L}_0)$.

(ii) $\forall x \in \mathrm{dcl}_{\mathcal{L}}(\mathcal{L}_0)\ \forall v \in \mathcal{L}_1\ (x \leqslant v \rightarrow \exists w \in \mathcal{L}_0\ (x \leqslant w \leqslant v))$ where $\mathrm{dcl}_{\mathcal{L}}(\mathcal{L}_0) = \{y \in \mathcal{L} \mid \exists x \in \mathcal{L}_0 (y \leqq x)\}$ *is the downward closure of $\mathcal{L}_0$ in $\mathcal{L}$.*

(iii) For every x in $\mathrm{dcl}_{\mathcal{L}}(\mathcal{L}_1)$, there is a largest $x_1 \in \mathcal{L}_1$, denoted by $\Pi_1(x)$, with $x_1 \leqslant x$ and a largest $x_0 \in \mathrm{dcl}_{\mathcal{L}}(\mathcal{L}_0)$, denoted by $\Pi_0(x)$ with $x_0 \leqslant x$ and moreover $x = \Pi_0(x) \vee \Pi_1(x)$.

(iv) If $x, y, z \in \mathrm{dcl}_{\mathcal{L}}(\mathcal{L}_1)$ and $x \vee y = z$ then $\Pi_0(z)$ and $\Pi_1(z)$ can be generated from the $\Pi_i(x)$ and $\Pi_i(y)$ by closing downward and under joins in $\mathcal{L}_1$ or in

$\mathrm{dcl}_{\mathscr{L}}(\mathscr{L}_0)$. To be more precise we first define the closure process $S_{L,\mathscr{L}}$ for any u.s.l.'s L and $\mathscr{L}$ (with possibly non-empty intersection) on subsets X of $L \cup \mathscr{L}$ by $S_{L,\mathscr{L}}(X) = \bigcup_n S^n_{L,\mathscr{L}}(X)$ where $S^0_{L,\mathscr{L}}(X) = X$ and

$$S^{n+1}_{L,\mathscr{L}}(X) = \{t \in L \cup \mathscr{L} \mid \exists r, s \in S^n_{L,\mathscr{L}}(X)[r, s, t \in L \;\&\; t \leqq_L r \vee s]$$
$$\text{or } \exists r, s \in S^n_{L,\mathscr{L}}(X)[r, s, t \in \mathscr{L} \;\&\; t \leqq_{\mathscr{L}} r \vee s]\}.$$

If we now set $S(X) = S_{\mathscr{L}_1, \mathrm{dcl}_{\mathscr{L}}(\mathscr{L}_0)}(X)$ we can state this requirement as

$$\Pi_0(z), \Pi_1(z) \in S(\{\Pi_0(x), \Pi_1(x), \Pi_0(y), \Pi_1(y)\}).$$

(Note that if L is finite (as it is here with $L = L_1$) then $S_{L,\mathscr{L}}(X) = S^n_{L,\mathscr{L}}(X)$ for some n since the sequence can continue to increase only by adding on new elements of L.)

Before constructing the "specially saturated" extension of our given $\mathscr{L}$, we prove some simple facts about special extensions that we will need later.

LEMMA 3.3. *Transitivity of* $\subseteq_{\mathrm{sp}}$. *If* $\mathscr{L}_0 \subseteq_{\mathrm{sp}} \mathscr{L}_1(\mathscr{L})$ *and* $\mathscr{L}_1 \subseteq_{\mathrm{sp}} \mathscr{L}_2(\mathscr{L})$ *then* $\mathscr{L}_0 \subseteq_{\mathrm{sp}} \mathscr{L}_2(\mathscr{L})$.

PROOF. (i) That $\mathscr{L}_0 \subseteq_{\mathrm{end}} \mathscr{L}_2$ is clear as is (ii) by applying it for both given extensions.

(iii) Let Π^1_i, Π^2_i be the projection functions for $\mathscr{L}_0 \subseteq \mathscr{L}_1$ and $\mathscr{L}_1 \subseteq \mathscr{L}_2$ respectively. To get the required functions for $\mathscr{L}_0 \subseteq \mathscr{L}_2$ we simply set $\Pi_0(x) = \Pi^1_0\Pi^2_0(x)$ and $\Pi_1(x) = \Pi^2_1(x)$. That $\Pi_1(x)$ is as required is clear. For Π_0 consider any $y \leqslant x$, $y \in \mathrm{dcl}\,\mathscr{L}_0$, $x \in \mathrm{dcl}\,\mathscr{L}_2$, $y \leqslant \Pi^2_0(x)$ and so $y \leqslant \Pi^1_0\Pi^2_0(x)$. Of course $x = \Pi_0(x) \vee \Pi_1(x)$ since $x = \Pi^2_0(x) \vee \Pi^2_1(x) = \Pi^1_0\Pi^2_0(x) \vee \Pi^1_1\Pi^2_0(x) \vee \Pi^2_1(x)$ and $\Pi^1_1\Pi^2_0(x) \leqslant \Pi^2_1(x) = \Pi_1(x)$ and $\Pi^1_0\Pi^2_0(x) = \Pi_0(x)$.

(iv) Given any $x \vee y = z$ in $\mathrm{dcl}\,\mathscr{L}_2$ argue by induction that for every t generated by S_2 for $\mathscr{L}_1 \subseteq_{\mathrm{sp}} \mathscr{L}_2$ from $X_2 = \{\Pi^2_i(x), \Pi^2_i(y)\}$ one gets t, if $t \in \mathscr{L}_2$, and $\Pi^1_i(t)$, if $t \in \mathrm{dcl}\,\mathscr{L}_1$, in the generation process S associated with $\mathscr{L}_0 \subseteq \mathscr{L}_2$ from $\{\Pi_1(x), \Pi_i(y)\} = X$. As $\Pi^2_i(z) \in S_2(x_2)$ this suffices to show that $\Pi_i(z) \in S(X)$. Begin with $S_{2,0}(X_2) = \{\Pi^2_i(x), \Pi^2_i(y)\}$. Of course it suffices to consider the $\Pi^2_i(x)$:

$$\Pi^2_1(x) = \Pi_1(x) \in S_0(X);$$

$$\Pi^1_0\Pi^2_0(x) = \Pi_0(x) \in S_0(X);$$

and $\Pi^1_1\Pi^2_0(x) \leqslant \Pi^2_1(x)$ and is in $\mathscr{L}_1 \subseteq \mathscr{L}_2$ and so in $S_1(X)$. Suppose we have $r, s \in S_{2,n}(X_2)$. If $r, s \in \mathscr{L}_2$, $t \in \mathscr{L}_2$ and $t \leqslant r \vee s$, then of course $t \in S(X)$ as $r, s \in S(X)$ by induction. If $r, s \in \mathrm{dcl}\,\mathscr{L}_1$ then $\Pi^1_1(r), \Pi^1_1(s)$ are in $S(X)$ by induction. As the generation process S_1 for $\mathscr{L}_0 \subseteq \mathscr{L}_1$ is contained in S,

$\Pi_i^1(r \vee s) \in S(X)$. Thus if $t \leq r \vee s$ then $\Pi_i^1(t) \leq \Pi_i^1(r \vee s)$ and so $\Pi_i^1(t) \in S(X)$ as well. $\square$

LEMMA 3.4. *Closure. If $\mathcal{L}_0 \subseteq_{sp} \mathcal{L}_1(\mathcal{L})$ and $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}$ with $\mathcal{L}_2$ finite then there is a finite $\mathcal{L}_3 \subseteq \mathcal{L}$ such that $\mathcal{L}_2 \subseteq \mathcal{L}_3$ and $\mathcal{L}_0 \subseteq_{sp} \mathcal{L}_1(\mathcal{L}_3)$.*

PROOF. Let Π_i be the given projection functions in $\mathcal{L}$. Let $\mathcal{L}_3$ be the (u.s.l.) closure of $\mathcal{L}_2 \cup \Pi_0[\mathcal{L}_2]$, i.e., those elements which are the join of one, x', in $\mathcal{L}_2$ and one, x'', in $\Pi_0[\mathrm{dcl}_{\mathcal{L}_2} \mathcal{L}_1] = \Pi_0[\mathcal{L}_2] \subseteq \mathrm{dcl}_{\mathcal{L}}(\mathcal{L}_0)$. Properties (i) and (ii) of the definition of $\mathcal{L}_0 \subseteq_{sp} \mathcal{L}_1(\mathcal{L}_3)$ hold for any $\mathcal{L}_3 \supseteq \mathcal{L}_1$ with $\mathcal{L}_3 \subseteq \mathcal{L}$ by $\mathcal{L}_0 \subseteq_{sp} \mathcal{L}_1(\mathcal{L})$. As $\mathcal{L}_3$ is finite one can define for $x \in \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_1$

$$\Pi_0'(x) = \max\{y \leq x \mid y \in \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_0\} \quad \text{and} \quad \Pi_1'(x) = \max\{y \leq x \mid y \in \mathcal{L}_1\} = \Pi_1(x).$$

We must show for (iii) that $x = \Pi_0'(x) \vee \Pi_1'(x)$. Say that $x \in \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_1$ and as above $x = x' \vee x''$. Of course $x'' \in \Pi_0[\mathcal{L}_2] \subseteq \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_0$ and so $x'' \leq \Pi_0'(x)$. $x' \in \mathrm{dcl}_{\mathcal{L}_2} \mathcal{L}_1$ and so $x' = \Pi_0(x') \vee \Pi_1(x')$ with $\Pi_0(x') \in \mathcal{L}_3$. Thus $\Pi_0(x') \leqq \Pi_0'(x)$. As $\Pi_1(x') \leqq \Pi_1'(x)$ as well we see that $x = x' \vee x'' = \Pi_0'(x) \vee \Pi_1'(x)$ as required.

Finally we must verify (iv). Let S' be the generation process for $\mathcal{L}_0 \subseteq \mathcal{L}_1$ $(\mathcal{L}_3)$ and S that for $\mathcal{L}_0 \subseteq \mathcal{L}_1$ $(\mathcal{L})$. Suppose $x \vee y = z$ in $\mathcal{L}_3$. Let $X' = \{\Pi_i'(x), \Pi_i'(y)\}$, $X = \{\Pi_i(x), \Pi_i(y)\}$. We need to show that $\Pi_i'(z) \in S'(\{\Pi_i'(x), \Pi_i'(y)\})$. We claim that $S(X) \cap \mathcal{L}_3 \subseteq S'(X')$. As $\Pi_i(z) \in S(X)$ which is downward closed in $\mathrm{dcl}_{\mathcal{L}} \mathcal{L}_0$ and in $\mathcal{L}_1$ and $\Pi_i'(z) \leq \Pi_i(z)$, this will clearly suffice. The first point is that $S(X) = S(X'')$ where

$$X'' = \{\Pi_i(x'), \Pi_i(x''), \Pi_i(y'), \Pi_i(y'')\}$$

and x', x'', y', y'' are chosen as in the definition of x and y being members of $\mathcal{L}_3$. As $\Pi_i(x')$, $\Pi_i(x'') \leq \Pi_i(x)$ and similarly for y, it is clear that $X'' \subseteq S(X)$ and so that $S(X'') \subseteq S(X)$. That $\Pi_i(x)$ and $\Pi_i(y) \in S(X'')$ follows from the facts that $x = x' \vee x''$ and $y = y' \vee y''$ via (iv) of $\mathcal{L}_0 \subseteq_{sp} \mathcal{L}_1(\mathcal{L})$. We now prove by induction that if $t \in S(X'') \cap \mathrm{dcl}_{\mathcal{L}} \mathcal{L}_0$ then $\exists t' \in S'(x') \cap \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_0 (t \leqq t')$ and that if $t \in S(X'') \cap \mathcal{L}_1$ then $t \in S'(X') \cap \mathcal{L}_1$. As $S'(X')$ is downward closed in $\mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_0$ this will prove the claim. For $n = 0$ note first that $\Pi_1(x')$, $\Pi_1(x'') \leq \Pi_1(x) = \Pi_1'(x) \in S'(X')$ and so $\Pi_1(x'), \Pi_1(x'') \in S'(X')$ and similarly for y. Next $\Pi_0(x') \leq \Pi_0'(x)$ and $\Pi_0(x'') \leq x'' \in \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_0$ and so $x'' \leq \Pi_0'(x)$ as well. Thus $\Pi_0'(x), \Pi_0'(x'') \in S'(x')$ and similarly for y.

Suppose now that $r, s \in S_n(X'') \cap \mathcal{L}_1$ and $t \leq r \vee s$, $t \in \mathcal{L}_1$. Then by induction $r, s \in S'(X')$ and so $t \in S'(X')$. Finally, if $r, s \in S_n(X'') \cap \mathrm{dcl}_{\mathcal{L}} \mathcal{L}_0$ then by induction there are $r', s' \in S'(X') \cap \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_0$ with $r \leqq r'$ and $s \leqq s'$. If $t \leqq r \vee s$ (and so $t \in S_{n+1}(X'')$) then $t \leqq r' \vee s' \in S'(X') \cap \mathrm{dcl}_{\mathcal{L}_3} \mathcal{L}_0$ as required. $\square$

LEMMA 3.5. *If $\mathcal{L}$ is a size $\aleph_1$ u.s.l. with 0 and the c.p.p. then it has an end extension $\mathcal{L}^*$ (of size $\aleph_1$ with 0 and the c.p.p.) which can be given as the union of a continuous monotonic sequence of downward closed sub u.s.l.'s $\mathcal{L}_\alpha$ which are* saturated, *i.e., for every finite sub u.s.l. L_0 of $\mathcal{L}_{\alpha+1}$ and every isomorphism type of a finite u.s.l. end extension of L_0 there is an $L_1 \subseteq \mathcal{L}_{\alpha+1}$ with $L_1 - L_0 \subseteq \mathcal{L}_{\alpha+1} - \mathcal{L}_\alpha$ which realizes this type over L_0 such that $L_0 \subseteq_{sp} L_1(\mathcal{L}_{\alpha+1})$. (For notational convenience we let $\mathcal{L}_{-1} = \emptyset$ and allow α to be -1 as well.)*

PROOF. We really only need to be able to construct extensions for one L_0 and isomorphism type at a time for we can then dovetail to get the desired result. Thus we first need a one step extension.

SUBLEMMA 3.6. *If L_0 is a finite sub u.s.l. of an $\mathcal{L}$ as above and a finite isomorphism type over L_0 is given, then there is an end extension $\mathcal{L}'$ of $\mathcal{L}$ (of size $\aleph_1$ with 0 and the c.p.p.) with an $L_1 \subseteq \mathcal{L}'$ realizing the given type over L_0 such that $L_0 \subseteq_{sp} L_1(\mathcal{L}')$.*

PROOF. We begin with any L_1 realizing the given type over L_0 with elements of $L_1 - L_0$ denoted by symbols not used in $\mathcal{L}$. We use $S = S_{L_1,\mathcal{L}}$ to define the elements of $\mathcal{L}'$,

$$\mathcal{L}' = \{S(X) \mid X \text{ a finite non-empty subset of } L_1 \cup \mathcal{L}\}.$$

The u.s.l. structure on $\mathcal{L}'$ is given by

$$S(X) \leqq S(Y) \Leftrightarrow S(X) \subseteq S(Y)$$

and

$$S(X) \vee S(Y) = S(X \cup Y).$$

One must now check that this defines an u.s.l. structure. Of course $S(\{0\}) = \{0\}$ is the 0 of $\mathcal{L}'$ and $\preccurlyeq$ defines a partial order. It is clear that $S(X), S(Y) \subseteq S(X \cup Y)$. Finally, if $S(X), S(Y) \subseteq S(Z)$ then $X, Y \subseteq S(Z)$ and so $S(X \cup Y) \subseteq S(Z)$ as required.

To formally guarantee that $\mathcal{L} \subseteq \mathcal{L}'$ we identify $S(\{x\})$ with x for $x \in \mathcal{L}$. Again we must check that this is an u.s.l. isomorphism. The point here is that for $X \subseteq \mathcal{L}$, $S(X) = \mathrm{dcl}_{\mathcal{L}}(\vee X)$ by definition of S and the fact that $L_0 \subseteq_{end} L_1$. Thus $x < y \Rightarrow S(\{x\}) \subseteq S(\{y\})$; $x \neq y \Rightarrow S(\{x\}) \neq S(\{y\})$; and $x \vee y = z \Rightarrow S(\{x\}) \vee S(\{y\}) = S(\{x, y\})$ is in fact $S(\{z\})$.

We must now verify that $\mathcal{L}'$ has all the required properties.

(i) $\mathcal{L} \subseteq_{end} \mathcal{L}'$: If $S(X) \subseteq S(\{y\})$, $y \in \mathcal{L}$ then $X \subseteq S(X) \subseteq \mathrm{dcl}_{\mathcal{L}}\{y\}$. Thus $X \subseteq \mathcal{L}$ and $S(X) = S(\{\vee X\})$.

(ii) $\mathscr{L}'$ has the c.p.p.: The point here is that for every X, $S(X)$ is countable. ($X = S_0(X)$ is finite and if $S_n(X)$ is countable then $S_{n+1}(X)$ only adds on elements of L_1 (which is finite) or ones of $\mathscr{L}$ below the join of two such in $S_n(X)$. As $\mathscr{L}$ has the c.p.p. this set is also countable.) As $S(Y) \leqq S(X) \Rightarrow Y \subseteq S(X)$ and Y is finite there can then be only countably many such elements.

(iii) $L_1^* = \{S\{x\} \mid x \in L_1\} \subseteq \mathscr{L}'$ realizes the same type over L_0 as does L_1:

(a) If $x < y$ in L_1 then $x \in S_1(\{y\})$ and so $S(\{x\}) \subseteq S(\{y\})$.

(b) If $x \vee y = z$ in L_1 then $x, y \in S_1(\{z\})$ and so $S(\{x\}) \vee S(\{y\}) = S(\{x, y\}) \leqq S(\{z\})$. On the other hand $z \in S(\{x, y\})$ and so $S(\{z\}) \leqq S(\{x\}) \vee S(\{y\})$.

(c) If $x \not\leqslant y$ are in L_1 we must show that $S(\{x\}) \not\subseteq S(\{y\})$. We claim that $S(\{y\}) = T = \mathrm{dcl}_{L_1}(\{y\}) \cup \{x \in \mathscr{L} \mid (\exists z \in L_0)\,(z < y \text{ in } L_1 \text{ and } x < z \text{ in } \mathscr{L})\}$. Now $y \in T \subseteq S_2\{y\}$ and so we need only show by induction that $S_n(\{y\}) \subseteq T$ for $n > 0$. The point here is first that if $r, s \in \mathscr{L} \cap T$ and $t \leqslant r \vee s$ then $t \in T$ and second that if $r, s \in L_1 \cap T$ and $t \leqslant r \vee s$ is in L_1 then $t \in T$. Thus if $x \not\leqslant y$ is in L_1, $x \notin S(\{y\})$.

We can now identify L_1^* with L_1.

(iv) $L_0 \subseteq_{\mathrm{sp}} L_1(\mathscr{L}')$: We verify the four clauses in Definition 3.2.

(a) $L_0 \subseteq_{\mathrm{end}} L_1(\mathscr{L}')$ since $\mathscr{L} \subseteq_{\mathrm{end}} \mathscr{L}'$.

(b) Suppose $S(X) \in \mathrm{dcl}_{\mathscr{L}'}(L_0) = \mathrm{dcl}_{\mathscr{L}}(L_0)$. Then $S(X) = S(\{x\})$ with $x \in \mathrm{dcl}_{\mathscr{L}} L_0$. If $S(\{x\}) \subseteq S(\{y\})$ with $y \in L_1$ then $x \in \mathrm{dcl}_{L_1}\{y\}$ or $(\exists z \in L_0)(z <_{L_1} y \;\&\; x <_{\mathscr{L}} z)$. The second possibility is exactly the one required. If, however, $x \in \mathrm{dcl}_{L_1}\{y\}$, then $x \in L_1$ (and $x \leqslant y$). As $x \leqslant u$ for some $u \in L_0$ as well and $L_0 \subseteq \mathscr{L} \subseteq_{\mathrm{end}} \mathscr{L}_1'$ and $L_1 \cap \mathscr{L} = L_0$, $x \in L_0$ and it already is the required element.

(c) Suppose $S(X) \in \mathrm{dcl}_{\mathscr{L}'}(L_1)$ so $S(X) \subseteq S(\{y\})$ for some $y \in L_1$. As L_1 is finite there is clearly a largest $x_1 \in L_1$ below $S(X)$. Set $\Pi_1(S(X)) = x_1 = S(\{x_1\})$. We have established above that

$$S(\{y\}) = \mathrm{dcl}_{L_1}(\{y\}) \cup \{x \in \mathscr{L} \mid (\exists z \in L_0)(z <_{L_0} y \;\&\; x <_{\mathscr{L}} z)\}.$$

Now argue by induction that for $n \geqq 1$, $S_n(X)$ is the union of some subset of L_1 and a finite number of downward cones in $\mathrm{dcl}_{\mathscr{L}}(L_0)$ (i.e., sets of the form $\{u \mid u \leqslant v\}$ for $v \in \mathrm{dcl}_{\mathscr{L}}(L_0)$). For $n = 1$ this follows from $X \subseteq S(\{y\})$ and so $X \subseteq L_1 \cup \mathrm{dcl}_{\mathscr{L}}(L_0)$. The rest is immediate from the definition of $S_{n+1}(X)$ in terms of $S_n(X)$. As we know that there is an n such that $S_{n+1}(X) = S_n(X)$, $S_n(X) - L_1$ must consist of a single such cone, say $\{u \in \mathscr{L} \mid u \leqslant x_0\}$ for some $x_0 \in \mathrm{dcl}_{\mathscr{L}}(L_0)$. $S\{x_0\} = x_0$ is then clearly the largest element of $\mathrm{dcl}_{\mathscr{L}}(L_0)$ below $S(X)$. As $\mathrm{dcl}_{\mathscr{L}'}(L_0) = \mathrm{dcl}_{\mathscr{L}}(L_0)$ we can set $\Pi_0(S(X)) = S(\{x_0\})$. Finally $S(X) \subseteq S(\{x_0, x_1\})$ since

$$S(X) = \{y \in L_1 \mid y \leqslant x_1\} \cup \{y \in \mathscr{L} \mid y \leqslant x_0\}.$$

Thus $S(X) = \Pi_0(S(X)) \vee \Pi_1(S(X))$.

(d) Suppose $S(X)$, $S(Y)$ and $S(Z) \in \mathrm{dcl}_{\mathscr{L}'}(L_1)$ and $S(X) \vee S(Y) = S(X \cup Y) = S(Z)$. Now $S(X) = S(\{x_0, x_1\})$, $S(Y) = S(\{y_0, y_1\})$ and so $S(\{x_0, x_1, y_0, y_1\}) = S(Z)$ but $z_0, z_1 \in S(Z)$ and so $z_0, z_1 \in S(\{x_0, x_1, y_0, y_1\})$ which is contained in $S_{L_1, \mathrm{dcl}_{\mathscr{L}'}(L_0)}(\{S\{x_0\}, S\{x_1\}, S\{y_0\}, S(\{x, y\})\})$ since $S(\{x_0, x_1, y_0, y_1\}) \subseteq L_1 \cup \mathrm{dcl}_{\mathscr{L}}(L_0) = L_1 \cup \mathrm{dcl}_{\mathscr{L}'}(L_0)$. □

We return now to the proof of the lemma. Consider any finite $L_0 \subseteq \mathscr{L}$ and an isomorphism type of L_1 over L_0. Form $\mathscr{L}'$ as in the sublemma and let $\mathscr{L}_{0,0}$ be the least downward closed sub u.s.l. of $\mathscr{L}'$ containing L_1 which is closed under Π_0 and Π_1. $\mathscr{L}_{0,0}$ clearly exists and is countable. Moreover by the closure under Π_0 and Π_1 it is easy to see that $L_0 \subseteq_{\mathrm{sp}} L_1(\mathscr{L}_{0,0})$. We can now choose another finite sublattice of $\mathscr{L}_{0,0}$ and another isomorphism type to generate an end extension $\mathscr{L}^2$ of $\mathscr{L}'$ as in the sublemma and so a countable end extension $\mathscr{L}_{0,1}$ of $\mathscr{L}_{0,0}$ in which the required extension exists and is special. As $\mathscr{L}_{0,0} \subseteq_{\mathrm{end}} \mathscr{L}_{0,1}$ extensions that are special in $\mathscr{L}_{0,0}$ (e.g. $L_0 \subseteq_{\mathrm{sp}} L_1(\mathscr{L}_{0,0})$) remain so in $\mathscr{L}_{0,1}$. [The point is that the definition of specialness depends only on $\mathrm{dcl}_{\mathscr{L}_{0,0}}(L_1)$ which is the same as $\mathrm{dcl}_{\mathscr{L}_{0,1}}(L_1)$.] By dovetailing over all finite sub u.s.l.'s and all possible types of finite extensions we can eventually get $\mathscr{L}^{(\omega)}$, an end extension of $\mathscr{L}$ with $\mathscr{L}_0 = \mathscr{L}_{0,\omega} \subseteq_{\mathrm{end}} \mathscr{L}^{(\omega)}$ so that $\mathscr{L}_0$ satisfies the requirements of the lemma. By continuing to dovetail (always using new elements) so as to include all elements of the $\mathscr{L}^{(\alpha)}$ as well we can get our desired $\mathscr{L}^*$ as $\mathscr{L}^{(\omega_1)}$ along with the division into countable $\mathscr{L}_\alpha$ as required by the lemma. □

By this lemma it suffices to consider only those u.s.l.'s $\mathscr{L}^*$ such that there is a continuous monotonic sequence $\mathscr{L}_\alpha$ of downward closed saturated countable sub u.s.l.'s with $\bigcup \mathscr{L}_\alpha = \mathscr{L}^*$. We fix such a system and will define our notions of forcing $\mathscr{P}_{\alpha+1}$ to be the conditions in the notion of forcing appropriate to $\mathscr{L}_{\alpha+1}$ which are represented by conditions in $\mathscr{G}_\alpha$ via special extensions. To be precise suppose we have a definition for a class $\mathscr{C}_5$ of dense sets in the notion of forcing appropriate to any countable u.s.l. analogous to that of Section 1.

DEFINITION 3.7. *The sequence of forcing notions.* Given $\mathscr{L}^* = \bigcup \mathscr{L}_\alpha$ as above we define $\mathscr{P}_\alpha$ and $\mathscr{G}_\alpha$ by simultaneous induction. $\mathscr{P}_0$ is the notion of forcing appropriate for $\mathscr{L}_0$ and $\mathscr{G}_0$ is any $\mathscr{C}_5$-generic filter on $\mathscr{P}_0$. Suppose $\mathscr{P}_\alpha$ is defined and $\mathscr{G}_\alpha$ is a $\mathscr{C}_5$-generic filter on $\mathscr{P}_\alpha$ (we will later verify that one exists). $\mathscr{P}_{\alpha+1}$ is the collection of all conditions P in the notion of forcing for $\mathscr{L}_{\alpha+1}$ such that there is a $P' \in \mathscr{G}_\alpha$ and a ϕ such that $\mathrm{dom}\,\phi \subseteq L_{P'}$, $\phi(P') = P$, $\phi \restriction (L_P \cap \mathscr{L}_\alpha) = \mathrm{id}$, $L_P \cap \mathscr{L}_\alpha \subseteq_{\mathrm{sp}} \phi^{-1}(L_P)(\mathscr{L}_\alpha)$ and $L_P \cap \mathscr{L}_\alpha \subseteq \phi^{-1}[L_P]$ satisfies the *rank condition* where we say that $L_0 \subseteq L_1$ satisfies the rank condition if $\forall x \in (L_1 - L_0)\, \forall y \in L_1$ $(\operatorname{rk} y \leqq \operatorname{rk} x)$. Of course for $z \in \mathscr{L}^*$, $\operatorname{rk} z = \mu\beta(z \in \mathscr{L}_\beta - \bigcup_{\gamma<\beta} \mathscr{L}_\gamma)$. $\mathscr{G}_{\alpha+1}$ is then any $\mathscr{C}_5$-generic filter on $\mathscr{P}_{\alpha+1}$. For limit ordinals λ we just set $\mathscr{P}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{P}_\alpha$ and $\mathscr{G}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{G}_\alpha$.

The crucial step now is to define the class of dense sets $\mathscr{C}_5$ needed to make $\mathscr{C}_5$-genericity of $\mathscr{G}_\alpha$ imply the existence of a $\mathscr{C}_5$-generic $\mathscr{G}_{\alpha+1} \subseteq \mathscr{P}_{\alpha+1}$. Again the density of the $D_{0,n}$ (totality), $D_{2,x,y}$ (diagonalization) (and $D_{4,e,x}$ if employed) in $\mathscr{P}_{\alpha+1}$ follow immediately from the corresponding genericity requirements on $\mathscr{G}_\alpha$. The density here of the $D_{3,e,x}$ (initial segments) will also follow from the corresponding requirements on $\mathscr{G}_\alpha$ because of the specialness of our representations. Thus the only real problem is to guarantee the extendibility lemma.

Suppose we are given a $P \in \mathscr{P}_{\alpha+1}$ represented by $P' \in \mathscr{G}_\alpha$ with $\phi(P') = P$ and some $x \notin L_P$. To add in x we must refine P to a condition Q with L_Q containing the u.s.l. L generated (in $\mathscr{L}_{\alpha+1}$) by $L_P \cup \{x\}$. To get such a Q we need an appropriate $Q' \in \mathscr{G}_\alpha$ with representatives for the type of this u.s.l. of the required form and a corresponding mapping ψ such that $\psi(Q') \leqq \phi(P')$.

Definition 3.8. *Amalgamation.* Let $\mathscr{P}$ be the notion of forcing appropriate to a countable u.s.l. $\mathscr{L}$. $\mathscr{C}_5'$ contains $\mathscr{C}_4$ and for each finite isomorphism type I of u.s.l.'s and maps as in the commutative diagram (3.8(i))

$$\begin{array}{ccc} \underline{L}_0 & \hookrightarrow_{\text{end}} & \underline{L}_1 \\ \downarrow & & \downarrow j \\ \underline{L}_0' & \hookrightarrow_{\text{end}} & \underline{L}_1' \end{array}$$

Fig. 3.8(i).

and every condition $R \in \mathscr{P}$ with a realization $g : \underline{L}_1 \to L_1 \subseteq L_R$ of $\underline{L}_1$ such that $g[\underline{L}_0] = L_0 \subseteq_{\text{sp}} L_1 = g[\underline{L}_1](\mathscr{L})$ and every realization $f : \underline{L}_0' \to L_0' \subseteq \mathscr{L}$ such that $f \restriction \underline{L}_0 = g \restriction \underline{L}_0$ the sets $D'_{5,I,R,g,f} = \{Q \mid Q$ is incompatible with R or there is an $h : \underline{L}_1' \to L_Q$ realizing this type with $h \restriction \underline{L}_0' = f \restriction \underline{L}_0'$ such that $h[\underline{L}_0'] \subseteq_{\text{sp}} h[\underline{L}_1']$ and $\phi(Q) \leqq R \restriction L_1$ where $\operatorname{rg} \phi = hj[\underline{L}_1]$, $\operatorname{dom} \phi = L_1$ and ϕ is given by $hjg^{-1}(x) \mapsto x$ for $x \in L_1\}$. The situation is pictured in Fig. 3.8(ii).

$\mathscr{C}_5'$ also includes for every $L_0 \subseteq_{\text{sp}} L_1(\mathscr{L})$ the sets $D_{5,L_0,L_1} = \{Q \mid L_0 \subseteq_{\text{sp}} L_1(L_Q)\}$.

The basic lemma, whose proof we postpone to Section 4 so as not to interfere overly much with the flow of the overall argument, is then the following:

Lemma 3.9. *If $\mathscr{P}$ is appropriate to $\mathscr{L}$ and $\mathscr{L}$ is saturated then the sets $D'_{5,I,R,g,f}$ and D_{5,L_0,L_1} are dense in $\mathscr{P}$.*

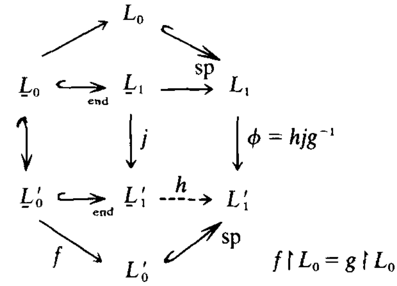

Fig. 3.8(ii).

Now let $\mathscr{L}^* = \bigcup_{\alpha<\omega_1} \mathscr{L}_\alpha$ be as in Lemma 3.5 and $\mathscr{P}_\alpha$, $\mathscr{G}_\alpha$ as in Definition 3.7.

To reflect the rank condition on representations for conditions in $\mathscr{P}_{\alpha+1}$ we modify the $\mathscr{C}'_5$ slightly in this setting to get $\mathscr{C}_5$ by requiring in the definition of the $D_{5,I,R,g,f}$ that $g[\underline{L}_0] \subseteq g[\underline{L}_1]$ and $h[\underline{L}'_0] \subseteq h[\underline{L}'_1]$ satisfy the rank condition. Of course for $\mathscr{L}_0$ the two notions coincide and so Lemma 3.9 gives us the $\mathscr{C}_5$-genericity of $\mathscr{G}_0$.

LEMMA 3.10. *If $\mathscr{G}_\alpha$ is $\mathscr{C}_5$-generic for $\mathscr{G}_\alpha$ then there is a $\mathscr{G}_{\alpha+1}$ which is $\mathscr{C}_5$-generic for $\mathscr{P}_{\alpha+1}$.*

PROOF. We must show that each of the required sets is dense in $\mathscr{P}_{\alpha+1}$. Consider any $P \in \mathscr{P}_{\alpha+1}$. Let $P' \in \mathscr{G}_\alpha$ and ϕ be as in the definition of $\mathscr{P}_{\alpha+1}$.

(a) $D_{0,n}$. Let $Q' \leqq P'$ be given by $\mathscr{C}_0$-genericity of $\mathscr{G}_\alpha$, i.e., $Q' \in D_{0,n}$. Clearly $Q = \phi(Q') \leqq \phi(P') = P$ and so $Q \in D_{0,n} \cap \mathscr{P}_{\alpha+1}$. □

(b) $D_{1,x}$. Let $\underline{L}_0$ be the type of $L_P \cap \mathscr{L}_\alpha = L_0$ and $\underline{L}_1 \supseteq \underline{L}_0$ that of $L_P \supseteq L_P \cap \mathscr{L}_\alpha$ with realization $k: \underline{L}_1 \to L_P$. Then we have a realization $g: \underline{L}_1 \to \phi^{-1}[L_P] = L_1 \subseteq L_{P'}$ with $g = \phi^{-1}k$ so that $g \restriction \underline{L}_0 = k$ (as $\phi \restriction L_0 = \mathrm{id}$). By the definition of $\mathscr{P}_{\alpha+1}$, $g[\underline{L}_0] = L_0 \subseteq_{\mathrm{sp}} L_1 = g[\underline{L}_1](\mathscr{L}_\alpha)$ and the rank condition is satisfied. Now apply the $\mathscr{C}_5$-genericity of $\mathscr{G}_\alpha$ to get a $Q' \leqq P'$ with $Q' \in \mathscr{G}_\alpha \cap D_{5,I,P',g,f}$ where I is given by specifying $\underline{L}'_0$ and $\underline{L}'_1$ as the types of $L \cap \mathscr{L}_\alpha$ and L where L is the u.s.l. generated (in $\mathscr{L}_{\alpha+1}$) by $L_P \cup \{x\}$, j as just inclusion and $f: \underline{L}'_0 \to L \cap \mathscr{L}_\alpha = L'_0$ as the restriction of the natural realization $k: \underline{L}'_1 \to L$ (which of course extends $k \restriction \underline{L}_1$ and sends $\underline{x}$ to x) to $\underline{L}'_0$.

Now let $\psi = kh^{-1}$ so that $\psi^{-1}[L \cap \mathscr{L}_\alpha] = hk^{-1}(L \cap \mathscr{L}_\alpha) = h[\underline{L}'_0] \subseteq_{\mathrm{sp}} h[\underline{L}'_1] = hk^{-1}[L] = \psi^{-1}[L]$. As $h[\underline{L}'_0] \subseteq h[\underline{L}'_1]$ also satisfies the rank condition, $\psi(Q') = Q \in \mathscr{P}_{\alpha+1}$ and $L_Q = L$ which of course contains x. By the definition of $P_{5,I,P',g,f}$ $\phi'(Q') \leqq P' \restriction L_1$ where $\mathrm{dom}\,\phi' = hj[\underline{L}_1]$ and $\phi': hj(\underline{x}) \mapsto \phi^{-1}(x)$. Now $\psi(h(\underline{x})) = kh^{-1}(h(\underline{x})) = k(\underline{x}) = x$ and so $\psi = \phi\phi'$ on $h[\underline{L}_1]$. Thus $Q = \psi(Q') \leqq \phi\phi'(Q') \leqq \phi(P') = P$. □

(c) $D_{2,e,x,y}$ for $x \not\leq y$. We may assume by (b) that $x, y \in L_P$. Choose $Q' \leqq P'$ with $Q' \in \mathcal{G}_\alpha \cap D_{2,e,\phi^{-1}(x),\phi^{-1}(y)}$. $Q = \phi(Q') \in \mathcal{P}_{\alpha+1}$ and as $Q' \Vdash \neg(\phi_e^{G_{\phi^{-1}(y)}} = G_{\phi^{-1}(x)})$, $Q = \phi(Q') \Vdash \neg(\phi_e^{G_y} = G_x)$. □

(d) $D_{3,e,x}$. Again we may assume that $x \in L_P$. Choose $Q' \leqq P'$ with $Q' \in \mathcal{G}_\alpha \cap D_{3,e,\phi^{-1}(x)}$. Thus for some $y \leqslant \phi^{-1}(x)$, $y \in L_Q$,

$$Q' \Vdash (\phi_e^{G_{\phi^{-1}(x)}} \text{ is not total or } \phi_e^{G_{\phi^{-1}(x)}} \equiv_T G_y).$$

As $y \leqslant \phi^{-1}(x)$ and $L_P \cap \mathcal{L}_\alpha \subseteq_{sp} \phi^{-1}(L_P)(\mathcal{L}_\alpha)$ there are $y_0 \in \mathrm{dcl}_{\mathcal{L}_\alpha}(L_P \cap \mathcal{L}_\alpha)$, say $y_0 \leqslant z \in L_P \cap \mathcal{L}_\alpha$, and $y_1 \in \phi^{-1}(L_P)$ such that $y = y_0 \vee y_1$. Now we may assume by extendibility at level α that $y_0 \in L_{Q'}$ and so

$$Q' \Vdash (\phi_e^{G_{\phi^{-1}(x)}} \text{ is not total or}$$

$$\phi_e^{G_{\phi^{-1}(x)}} \equiv_T F_{Q',z,y_0}(G_z) \oplus G_{y_1}).$$

Thus, as $\phi(z) = z$,

$$\phi(Q') \Vdash (\phi_e^{Q_x} \text{ is not total or}$$

$$\phi_e^{G_x} \equiv_T F_{Q',z,y_0}(G_z) \oplus G_{\phi(y_1)}).$$

Of course $\phi(Q') \in \mathcal{P}_{\alpha+1}$ and $\phi(Q') \leqq P$. By (b) we may choose $Q \leqq \phi(Q')$, $Q \in \mathcal{P}_{\alpha+1}$ with $y_0 \in L_Q$. As $F_{Q',z,y_0} = F_{Q,z,y_0}$,

$$Q \Vdash (\phi_e^{G_x} \text{ is not total or}$$

$$\phi_e^{G_x} \equiv G_{y_0} \oplus G_{\phi(y_1)}).$$

There is, of course, a $v \in L_Q$ with $v = y_0 \vee \phi(y_1)$ and $Q \Vdash G_{y_0} \oplus G_{\phi(y_1)} \equiv_T G_v$ and so $Q \Vdash (\phi_e^{Q_x}$ is not total or $\phi_e^{G_x} \equiv_T G_v)$. Now by (c) we may even omit the second alternative unless $v \leqslant x$ as required. □

(e) D_{5,L_0,L_L}. By Lemma 3.4 there is a finite L such that $L_P \subseteq L \subseteq \mathcal{L}_{\alpha+1}$ and $L_0 \subseteq_{sp} L_1(L)$. Now argue exactly as in (b). □

(f) $D_{5,I,R,g,f}$. we need only consider the case that P refines R. By (b) we may assume that $L_0' = f[\underline{L}_0'] \subseteq L_P$. By (e) we may also assume that $L_0 \subseteq_{sp} L_1(L_P)$. Consider now the type of an extension L of L_P containing an L_1' with $L_1' - L_0' \subseteq L - L_P$ realizing the type specified in I over L_0' such that no extraneous ordering relations are introduced, i.e.,

$$(*) \qquad \forall x \in L_P[\exists y \in L_1'(x \leqslant y) \rightarrow x \in L_0']$$

and

$$(**) \qquad \forall x \in L[\exists y \in L_1'(x \leqslant y) \rightarrow x \in L_1'].$$

By the saturation of $\mathscr{L}_{\alpha+1}$ we may choose an $L \subseteq \mathscr{L}_{\alpha+1}$ with $L - L_P \subseteq \mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha$ realizing this type over L_P.

A picture of the lattices is given in Fig. 3.10(i) and the associated commutative diagram in Fig. 3.10(ii).

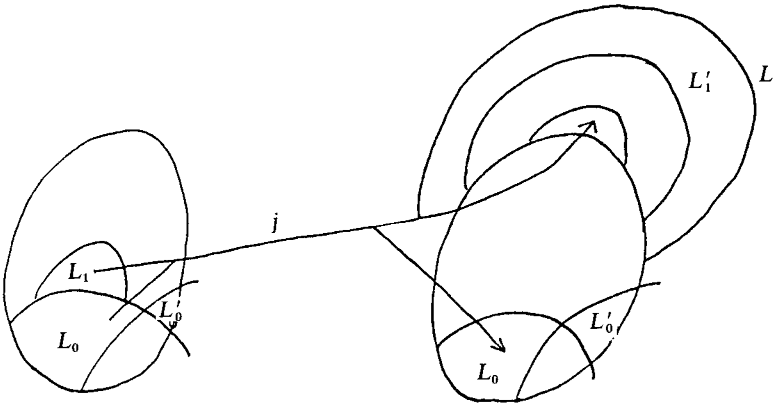

Fig. 3.10(i).

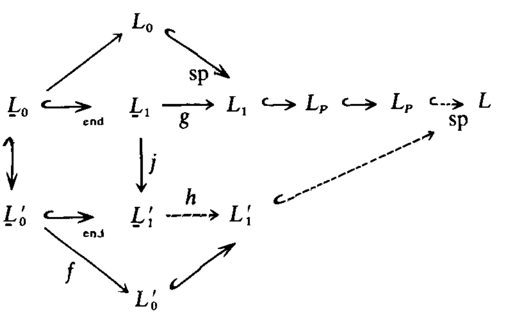

Fig. 3.10(ii).

We need a $Q' \leqq P'$ which will represent a $Q \in \mathscr{P}_{\alpha+1}$ satisfying all the requirements of $D_{5,\underline{I},R,g,f}$. We choose a $Q' \in \mathscr{G}_\alpha \cap D_{5,\tilde{I},P',\tilde{g},\tilde{f}}$ where $\tilde{\underline{L}}_0 = \underline{L}_0$; $\underline{L}_1 = \underline{L}_1$;

$$\tilde{g} : \tilde{\underline{L}}_0 \to \phi^{-1}[L_0] \qquad \text{via } \phi^{-1}g;$$

$$\tilde{g} : \tilde{\underline{L}}_1 \to \phi^{-1}[L_1] \qquad \text{via } \phi^{-1}g;$$

$\tilde{\underline{L}}_0'$ is the type of L_P; $\tilde{\underline{L}}_1'$ is the type of L; we extend g in the natural way extending $f \upharpoonright \underline{L}_0'$ to a realization $g : \tilde{\underline{L}}_1' \to L$ and so $g : \tilde{\underline{L}}_0' \to L_P$; we set $\tilde{f} : \tilde{\underline{L}}_0' \to \phi^{-1}[L_P]$ to be $\phi^{-1}g$; and $\tilde{j} : \tilde{\underline{L}}_1 \to \tilde{\underline{L}}_1'$ is just j as $\tilde{\underline{L}}_1 = \underline{L}_1$ and $j[\underline{L}_1] \subseteq \underline{L}_1' \subseteq \tilde{\underline{L}}_1'$. The diagrams for this set up are given in Figs. 3.10(iii), (iv).

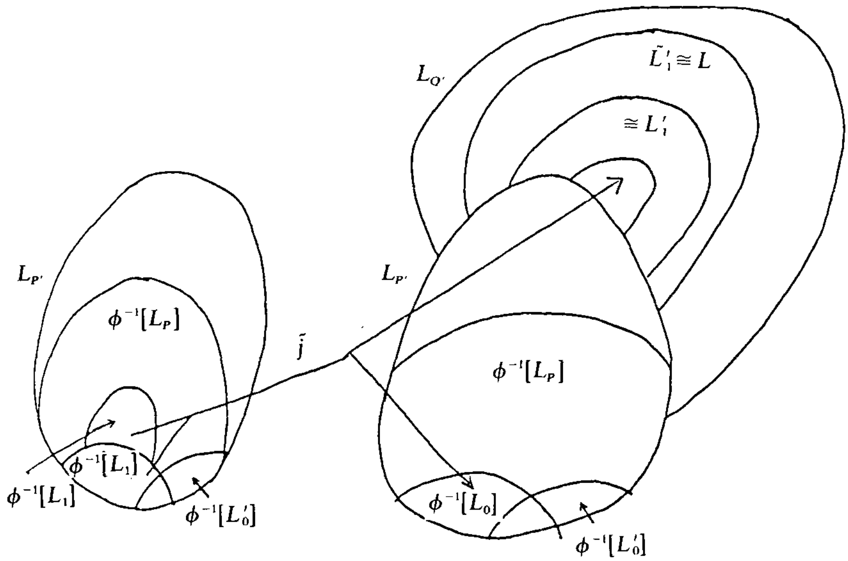

Fig. 3.10(iii).

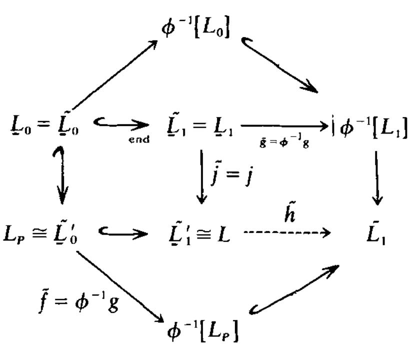

Fig. 3.10(iv).

We must first verify that $\tilde{g}$ has the properties required to apply $D_{5,\tilde{I},P',\tilde{g},\tilde{f}}$. As for the rank condition on $\tilde{g}[\tilde{\underline{L}}_0] \subseteq \tilde{g}[\tilde{\underline{L}}_1]$, i.e., for $\phi^{-1}[L_0] \subseteq \phi^{-1}[L_1]$, consider any $x \in \phi^{-1}[L_1] - \phi^{-1}[L_0] = \phi^{-1}[L_1 - L_0]$ and any $y \in \phi^{-1}[L_1]$. If $\operatorname{rk} x < \operatorname{rk} y$ then by the rank condition on $\phi^{-1}[L_P \cap \mathcal{L}_\alpha] \subseteq \phi^{-1}[L_P]$ we know that $x \notin \phi^{-1}[L_P] - \phi^{-1}[L_P \cap \mathcal{L}_\alpha]$. Thus $x \in \phi^{-1}[L_P \cap \mathcal{L}_\alpha]$ and so $\phi(x) = x$ and $x \in L_1 - L_0$. By the rank condition on $L_0 \subseteq L_1$ assumed in $D_{5,I,R,g,f}$, $\operatorname{rk} \phi(y) \leqq \operatorname{rk} x \leqq \alpha$ and so $\phi(y) \in \mathcal{L}_\alpha$ and $\phi(y) = y$. Thus $\operatorname{rk} y \leqq \operatorname{rk} x$ for the required contradiction.

We next verify that $\phi^{-1}[L_0] = \tilde{g}[\tilde{L_0}] \subseteq_{sp} \tilde{g}[\tilde{L_1}] = \phi^{-1}[L_1](\mathscr{L}_\alpha)$.

(i) As $L_0 \subseteq_{sp} L_1$, $\phi^{-1}[L_0] \subseteq_{end} \phi^{-1}[L_1]$.

(ii) Consider any $x \in \mathrm{dcl}_{\mathscr{L}_\alpha} \phi^{-1}[L_0]$ and $v \in \phi^{-1}[L_1]$ with $x \leqslant v$. If $v \in L_1 \cap \mathscr{L}_\alpha$ and $x \leqslant z \in \phi^{-1}[L_0]$ then by $L_P \cap \mathscr{L}_\alpha \subseteq_{sp} \phi^{-1}[L_P]$ there is a $t \in L_P \cap \mathscr{L}_\alpha$ such that $x \leqslant t \leqslant z$. Thus $x \leqslant t = \phi(t) \leqslant \phi(z) \in L_0$. By $L_0 \subseteq_{sp} L_1$ we then have $\phi(w) \in L_0$ with $x \leqslant \phi(w) \leqslant \phi(v) = v$. Of course $w = \phi(w)$ as $\phi(w) \leqslant v \in L_P \cap L_\alpha$. Thus $x \leqslant w \leqslant v$ and $w \in \phi^{-1}[L_0]$ as required. Now suppose that $v \in L_1 - \mathscr{L}_\alpha$. As $L_P \cap \mathscr{L}_\alpha \subseteq_{sp} \phi^{-1}[L_P](\mathscr{L}_\alpha)$, $x = x_0 \vee x_1$ with $x_0 \in \mathrm{dcl}_{\mathscr{L}_\alpha}(L_P \cap \mathscr{L}_\alpha)$ and $x_1 \in \phi^{-1}[L_P]$. As $x_0 \leqslant v$ there is a $u \in L_P \cap \mathscr{L}$ with $x_0 \leqslant u \leqslant v$ (by $L_P \cap \mathscr{L}_\alpha \subseteq_{sp} \phi^{-1}[L_P](\mathscr{L}_\alpha)$). Thus $u = \phi(u) \leqslant \phi(v) \in L_1$ and $u = \phi(u) = u_0 \vee u_1$ where $u_0 \in \mathrm{dcl}_{L_P} L_0$ and $u_1 \in L_1$ (by $L_0 \subseteq_{sp} L_1(L_P)$). As $\phi(u_0) = u_0 \leqslant \phi(v)$ and $u_0 \in \mathrm{dcl}_{L_P} L_0$ there is a $\phi(t) \in L_0$ with $u_0 \leqslant \phi(t) \leqslant \phi(v)$. Thus we have a $t \in \phi^{-1}[L_0]$ with $u_0 \leqslant t \leqslant v$. As $u_1 \leqslant u \leqslant v$, $\phi(u_1) = u_1 \leqslant \phi(u) = u \leqslant \phi(v)$. Thus by the rank condition for $L_0 \subseteq L_1$, $\phi(u_1) \in L_0$, i.e., $u_1 \in \phi^{-1}[L_0]$. Thus $x_0 \leqslant u \leqslant t \vee u_1 \leqslant v$ and $t \vee u_1 \in \phi^{-1}[L_0]$.

Next consider $x_1 \in \phi^{-1}[L_P]$. $\phi(x_1) \leqslant \phi(v)$ and $\phi(x_1) \in \mathrm{dcl}_{L_P} L_0$ (as $x_1 \leqslant x \in \mathrm{dcl}\, \phi^{-1}[L_0]$). By $L_0 \subseteq_{sp} L_1(L_P)$ we have a $\phi(s) \in L_0$ with $\phi(x_1) \leqslant \phi(s) \leqslant \phi(v)$. Now, of course, $x_1 \leqslant s \leqslant v$, $s \in \phi^{-1}[L_0]$ and we have $x = x_0 \vee x_1 \leqslant t \vee u_1 \vee s \leqslant v$ with $t \vee u_1 \vee s \in \phi^{-1}[L_0]$.

(iii) Consider any $x \in \mathrm{dcl}_{\mathscr{L}_\alpha} \phi^{-1}[L_1]$, say $x \leqslant v \in \phi^{-1}[L_1]$. By the choice of ϕ, $x = x_0 \vee x_1$ where

$$x_0 = \max\{y \leqslant x \mid y \in \mathrm{dcl}_{\mathscr{L}_\alpha} \phi^{-1}[L_P \cap \mathscr{L}_\alpha]\}$$

and

$$x_1 = \max\{y \leqslant x \mid y \in \phi^{-1}[L_P]\}.$$

By clause (ii) of the definition of the specialness of the extension there is a $w \in L_P \cap \mathscr{L}_\alpha$ with $x_0 \leqslant w \leqslant v$. Now $\phi(w) = w \leqslant \phi(v) \in L_1$ and so by $L_0 \subseteq_{sp} L_1(L_P)$, $w = w_0 \vee w_1$ with $w_0 \in \mathrm{dcl}_{L_P} L_0$ and $w_1 \in L_1$. As $w_0, w_1 \in \mathscr{L}_\alpha$, $\phi(w_0) = w_0$, $\phi(w_1) = w_1$ and so $w_0, w_1 \in \mathrm{dcl}_{\mathscr{L}_\alpha} \phi^{-1}[L_0]$. Thus $x_0 \leqslant w_0 \vee w_1$ is in $\mathrm{dcl}_{\mathscr{L}_\alpha} \phi^{-1}[L_0]$. Now as $x_1 \leqslant x \leqslant v$, $\phi(x_1) \leqslant \phi(v) \in L_1$. So we next consider $\phi(x_1) \in L_P \cap \mathrm{dcl}_{L_P} L_1$. Thus $\phi(x_1) = u_0 \vee u_1$ where $u_0 \in \mathrm{dcl}_{L_P}(L_0)$ and $u_1 \in L_1$. We then see that $x_1 = \phi^{-1}(u_0) \vee \phi^{-1}(u_1)$ with $\phi^{-1}(u_0) \in \mathrm{dcl}\, \phi^{-1}[L_0]$ and $\phi^{-1}(u_1) \in \phi^{-1}[L_1]$. We can thus try to define $\Pi_0(x) = x_0 \vee \phi^{-1}(u_0)$ and $\Pi_1(x) = \phi^{-1}(u_1)$ as $x = \Pi_0(x) \vee \Pi_1(x)$.

As x_1 is the largest element of $\phi^{-1}[L_P]$ below x and $\phi^{-1}(u_1)$ is the largest element of $\phi^{-1}[L_1]$ below x_1 it is clear that $\phi^{-1}(u_1)$ is the largest element of $\phi^{-1}[L_1]$ below x. We must show that $\Pi_0(x) = \max\{y \leqslant x \mid y \in \mathrm{dcl}_{\mathscr{L}_\alpha} \phi^{-1}[L_0]\}$.

Consider any relevant y. $y = y_0 \vee y_1$ with $y_0 \in \mathrm{dcl}_{\mathcal{L}_\alpha}(L_P \cap \mathcal{L}_\alpha)$, $y_1 \in \phi^{-1}[L_P]$. By definition of x_0, $y_0 \leqslant x_0$. Similarly $y_1 \leqslant x_1$ and so $\phi(y_1) \leqslant \phi(x_1)$. As $\phi(y_1) \in \mathrm{dcl}_{L_P} L_0$, $\phi(y_1) \leqslant u_0$. Thus $y = y_0 \vee y_1 \leqq x_0 \vee \phi^{-1}(u_0) = \Pi_0(x)$.

(iv) Let Π_i^1, S_1 and Π_i^2, S_2 (for $i = 0, 1$) be the projection functions and generating processes given by $L_0 \subseteq_{\mathrm{sp}} L_1(L_P)$ and $L_P \cap \mathcal{L}_\alpha \subseteq_{\mathrm{sp}} \phi^{-1}(L_P)(\mathcal{L}_\alpha)$ respectively. Thus $\Pi_i^3 = \phi^{-1}\Pi_i^1\phi$ and S_3 give (by the isomorphism) functions which witness $\phi^{-1}[L_0] \subseteq_{\mathrm{sp}} \phi^{-1}[L_1](\phi^{-1}[L_P])$. We can now write the functions Π_i witnessing $\phi^{-1}[L_0] \subseteq_{\mathrm{sp}} \phi^{-1}[L_1](\mathcal{L}_\alpha)$ as $\Pi_0(x) = \Pi_0^2(x) \vee \Pi_0^3\Pi_1^2(x)$ and $\Pi_1(x) = \Pi_1^3\Pi_1^2(x)$. Consider any $x \vee y = z$ in $\mathrm{dcl}_{\mathcal{L}_\alpha} \phi^{-1}[L_1]$. We know that if we apply S_2 to $X_2 = \{\Pi_0^2(x), \Pi_1^2(x), \Pi_0^2(y), \Pi_1^2(y)\}$ we eventually get $\Pi_0^2(z)$ and $\Pi_1^2(z)$. We claim by induction that applying S_2 to $X = \{\Pi_0(x), \Pi_1(x), \Pi_0(y), \Pi_1(y)\}$ we get every element of $\mathrm{dcl}_{\mathcal{L}_\alpha} L_P \cap \mathcal{L}_\alpha$ generated in $S_2(X_2)$ and for every element $t \in \phi^{-1}[L_P]$ in $S_2(x_2)$ we get $\Pi_0^3(t)$ and $\Pi_1^3(t)$. This, of course, implies that we get $\Pi_0^2(z)$, $\Pi_0^3(\Pi_1^2(z))$ and $\Pi_1^3(\Pi_1^2(z))$ in $S(x)$ and so $\Pi_0(z)$ and $\Pi_1(z)$ as required. The claim holds at level 0 by the definition of the Π_i. Suppose $r, s \in S_{2,n}(X_2) \cap \mathrm{dcl}_{\mathcal{L}_\alpha}(L_P \cap \mathcal{L})$. By induction $r, s \in S(x)$. The argument given in (ii) above for x_0 shows that $r, s \in \mathrm{dcl}_{\mathcal{L}_\alpha} \phi^{-1}[L_0]$ and so also for any $t \leqslant r \vee s$. Thus any t put into $S_{2,n+1}$ by the first clause of the definition is also put into $S(X)$. Finally suppose $r, s \in S_{2,n}(X) \cap \phi^{-1}[L_P]$ and $t \leqslant r \vee s$, $t \in \phi^{-1}[L_P]$. By induction $\Pi_i^3(r)$ and $\Pi_i^3(s) \in S(X)$. As the Π_i^3 witness $\phi^{-1}[L_0] \subseteq_{\mathrm{sp}} \phi^{-1}[L_1](\phi^{-1}[L_P])$ and the generation process S_3 is clearly contained inside that of S (for elements in $\mathrm{dcl}\, \phi^{-1}[L_1]$ as all of these are), $\Pi_i^3(r \vee s) \in S(X)$. As $\Pi_i^3(t) \leqslant \Pi_i^3(r \vee s)$ while $S(X) \cap \phi^{-1}[L_1]$ and $S(X) \cap \mathrm{dcl}_{\phi^{-1}[L_P]} \phi^{-1}[L_0]$ are downward closed (in $\phi^{-1}[L_1]$ and $\mathrm{dcl}_{\phi^{-1}[L_P]} \phi^{-1}[L_0]$ respectively), $\Pi_i^3(t) \in S(x)$ as required.

We can thus get our desired $Q' \in \mathcal{G}_\alpha \cap D_{5,\tilde{I},P',\tilde{g},\tilde{f}}$. We next define $\psi : L_{Q'} \to L$ by $\psi = g\tilde{h}^{-1} \upharpoonright \tilde{h}[\tilde{\underline{L}}_1']$ and let $Q = \psi(Q')$ so that $L_Q = L$. To see that $Q \in \mathcal{P}_{\alpha+1}$ we must verify that $\psi^{-1}[L_Q \cap \mathcal{L}_\alpha] = L_Q \cap \mathcal{L}_\alpha \subseteq_{\mathrm{sp}} \psi^{-1}[L_Q]$. Now $L_Q \cap \mathcal{L}_\alpha = L_P \cap \mathcal{L}_\alpha$ by our choice of L. Consider any $x \in L_P \cap \mathcal{L}_\alpha : \psi^{-1}(x) = \tilde{h}g^{-1}(x)$. $g^{-1}(x) \in \tilde{\underline{L}}_0'$ and $\tilde{h} \upharpoonright \tilde{\underline{L}}_0' = \tilde{f} = \phi^{-1}g$. Thus $\psi^{-1}(x) = \tilde{h}g^{-1}(x) = \phi^{-1}gg^{-1}(x) = \phi^{-1}(x) = x$. Next note that $\psi^{-1}[L_Q \cap \mathcal{L}_\alpha] = L_P \cap \mathcal{L}_\alpha \subseteq_{\mathrm{sp}} \phi^{-1}[L_P] = \tilde{h}[\tilde{\underline{L}}_0'] \subseteq_{\mathrm{sp}} \tilde{h}[\tilde{\underline{L}}_1'] = \tilde{h}g^{-1}[L] = \psi^{-1}[L_Q]$. Thus by the transitivity of $\subseteq_{\mathrm{sp}}$ (Lemma 3.3), $\psi^{-1}[L_Q \cap \mathcal{L}_\alpha] \subseteq_{\mathrm{sp}} \psi^{-1}[L_Q]$.

This argument shows that for $x \in L_P$, $\psi^{-1}(x) = \phi^{-1}(x)$ and so $\phi \subseteq \psi$. As $Q' \leqq P'$, $Q = \psi(Q') \leqq \phi(P') = P$. All that remains is to define h to show that $Q \in D_{5,I,R,g,f}$. We simply set $h = \psi\tilde{h} \upharpoonright \underline{L}_1' = g \upharpoonright \underline{L}_1'$ (as $L_1' \subseteq L$, $\underline{L}_1' \subseteq \tilde{\underline{L}}_1'$). As $g \supseteq f$, $h \upharpoonright \underline{L}_0' = f \upharpoonright \underline{L}_0'$. Next if $\theta : hj[\underline{L}_1] \to g[\underline{L}_1]$ is given by $hj(g^{-1}(x)) \mapsto x$ for $x \in L_1$, i.e., $gj(g^{-1}(x)) \mapsto x$, then we claim that $\theta(Q) \leqq P \upharpoonright L_1$. It, of course, suffices to check that for the map θ' sending $\psi^{-1}hjg^{-1}(x) \mapsto \phi^{-1}(x) = \psi^{-1}(x)$, $\theta'(Q') \leqq Q' \upharpoonright \phi^{-1}[L_1]$. But $\psi^{-1}hjg^{-1}(x) = \tilde{h}g^{-1}gjg^{-1}(x) = \tilde{h}jg^{-1}(x) = \tilde{h}j\tilde{g}^{-1}\phi^{-1}(x)$. Thus if

$y = \phi^{-1}(x) \in \tilde{g}[\tilde{L}_1]$ the map θ' is given by $\tilde{h}j\tilde{g}^{-1}(y) \mapsto y$. This however is the very map for which the definition of $D_{5,P',\tilde{l},\tilde{g},\tilde{f}}$ says that $\theta'(Q') \leqq Q' \upharpoonright \phi^{-1}[L_1]$. Finally $h[\tilde{L}_0'] = L_0'$ and $h[\tilde{L}_1'] = L_1'$ and we need only check that $L_0' \subseteq_{\mathrm{sp}} L_1'$ and that the rank condition is satisfied. Now we chose $L_0' \subseteq L_P \subseteq_{\mathrm{sp}} L$ with $L_1' \cap L_P = L_0'$ and so $L_0' \subseteq_{\mathrm{end}} L_1'$. If $x \in \mathrm{dcl}_{\mathscr{L}_{\alpha+1}} L_1'$, $x = x_0 \vee x_1$ with $x_0 = \max\{y \leqslant x \mid y \in \mathrm{dcl}_{\mathscr{L}_{\alpha+1}} L_P\}$ and $x_1 = \max\{y \leqslant x \mid y \in L\}$. By (*), $x_0 \in \mathrm{dcl}_{\mathscr{L}_{\alpha+1}} L_0' \subseteq \mathrm{dcl}_{\mathscr{L}_{\alpha+1}} L_P$ and so x_0 is the largest element of $\mathrm{dcl}_{\mathscr{L}_{\alpha+1}} L_0'$ below x. By (**), $x_1 \in L_1' \subseteq L$ and so x_1 is the largest element of that set below x. Similarly the generation process in $\mathrm{dcl}_{\mathscr{L}_{\alpha+1}} L_P$ and L applied to such decompositions gives the same results as the one for $\mathrm{dcl}_{\mathscr{L}_{\alpha+1}} L_0'$ and L_1'. Of course the other requirement (ii) is guaranteed by the corresponding one for $L_P \subseteq_{\mathrm{sp}} L(\mathscr{L}_{\alpha+1})$ and (*). Thus $L_0' \subseteq_{\mathrm{sp}} L_1'(\mathscr{L}_{\alpha+1})$. As $L_1' - L_0' \subseteq L - L_P \subseteq \mathscr{L}_{\alpha+1} - \mathscr{L}_\alpha$, the rank condition is fulfilled as well. □

All that remains is to note that at limits $\mathscr{C}_5$-genericity is automatic.

Lemma 3.11. *If $\mathscr{G}_\alpha$ is $\mathscr{C}_5$-generic in $\mathscr{P}_\alpha$ for every $\alpha < \lambda$ then $\mathscr{G}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{G}_\alpha$ is $\mathscr{C}_5$-generic in $\mathscr{P}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{P}_\alpha$.*

Proof. Note that the sequence is monotonic as $\mathscr{G}_\alpha \subseteq \mathscr{G}_{\alpha+1}$ — any $P \in \mathscr{G}_\alpha$ represents itself in $\mathscr{P}_{\alpha+1}$. Thus all the requirements for the $\mathscr{C}_5$-genericity of $\mathscr{G}_\lambda$ are guaranteed by the $\mathscr{C}_5$-genericity of each $\mathscr{G}_\alpha$, $\alpha < \lambda$.

Thus modulo the proofs of Lemmas 2.9 and 3.9 which will come in the next section we have completed the proof of our main result:

Theorem 3.12. *Every size $\aleph_1$ u.s.l. $\mathscr{L}$ with 0 and the c.p.p. is isomorphic to an initial segment of $\mathscr{D}$.*

Proof. Form $\mathscr{L}^* = \bigcup \mathscr{L}_\alpha$ an end extension of $\mathscr{L}$ as in Lemma 3.5. Then define $\mathscr{P}_\alpha$ and $\mathscr{C}_5$-generic $\mathscr{G}_\alpha$ as described above. The map sending $x \in \mathscr{L}^*$ to $\deg(G_x)$ where

$$G_x = \bigcup \{T_{P,x}(\varnothing) \mid \exists \alpha (P \in \mathscr{G}_\alpha \;\&\; x \in L_P)\}$$

defines an isomorphism of $\mathscr{L}^*$ onto an initial segment of $\mathscr{D}$ by the usual arguments from $\mathscr{C}_5$-genericity as in Theorem 1.17. As $\mathscr{L} \subseteq_{\mathrm{end}} \mathscr{L}^*$ the restriction of this map to $\mathscr{L}$ gives the desired embedding of $\mathscr{L}$ onto an initial segment of $\mathscr{D}$. □

Remark 3.13. Of course if we include the dense sets of $\mathscr{C}_4$ (modulo a translation to set coding) we guarantee that for each $x \in \mathscr{L}$ and $e \in \omega$ if $\phi_e^{G_x}$ is total then ϕ_e^X is total for every X on some recursive tree containing G_x as a branch. Thus each set T-reducible to G_x is in fact tt reducible to it. This then gives an embedding of $\mathscr{L}$ simultaneously onto an initial segment of the *tt*, *wtt* and T degrees.

4. Refinements and amalgamation

We can now revert to the notation of Section 2 so that $\mathscr{P}$ is the notion of forcing there defined for some given countable u.s.l. (with 0) $\mathscr{L}$. Our basic task in this section is to prove that one can construct the refinements of sequential tables needed to prove Lemmas 2.9 (extendibility of conditions) and 3.9 (amalgamation). Many of the ingredients of the construction of course come from Lerman [9], Lachlan and Lebeuf [8] or Lerman [10, appendix B].

THEOREM 4.1. *Suppose we are given u.s.l.'s* $\mathscr{L}_0 \subseteq_{sp} \mathscr{L}_1(\mathscr{L}_4)$, $\mathscr{L}_1 \subseteq \mathscr{L}_3 \subseteq \mathscr{L}_4$, $\mathscr{L}_2 \subseteq \mathscr{L}_4$, *an isomorphism* $l : \mathscr{L}_1 \xrightarrow{\sim} \mathscr{L}_2$ *with* $l \restriction \mathscr{L}_0 = \mathrm{id}$ (*see Diagram* 4.2) *such that any element* x *of* $\mathscr{L}_2$ *which is below any* $y \in \mathscr{L}_3$ *is in fact in* $\mathscr{L}_0$ *and a recursive extendible sequential table* Θ *for* $\mathscr{L}_3$. *We can then construct a recursive extendible sequential table* Φ *for* $\mathscr{L}_4$ *and recursive functions* k, g *and* F *such that for every* $i \in \omega$

(i) g *sends* Φ_i *to a positive u.s.l. table* $g\Phi_i$ *for* $\mathscr{L}_4$ *with* $g\alpha(x) = \alpha(\Pi_0(x) \vee l\Pi_1(x))$ *for* $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_1$ *and otherwise for each* $x \in \mathscr{L}_4$, $\alpha \in \Phi_i$, $g\alpha(x)$ *is a distinct element not appearing in* Φ_i (*i.e., not in the range of any element of* Φ_i).

(ii) F *is an isomorphism of positive tables for* $\mathscr{L}_4$ *such that* $F_x = \mathrm{id}$ *for* $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$ *and otherwise for each* $x \in \mathscr{L}_4$ *and* $g\alpha(x)$ (*for* $\alpha \in \Phi_i$), $F_x(g\alpha)(x)$ *is a distinct element not appearing in* Φ_i *or* $g\Phi_i$.

(iii) $\Phi_i \cup Fg\Phi_i \to (\Phi_i \cup Fg\Phi_i) \restriction \mathscr{L}_3 \hookrightarrow_a \Theta_{k(i)}$.

(iv) $\Phi_i \to \Phi_i \restriction \mathscr{L}_3 \hookrightarrow_a \Theta_{k(i)}$.

(v) $Fg\Phi_i \to Fg\Phi_i \restriction \mathscr{L}_3 \to \Theta_{k(i)}$.

Before giving the rather technical proof of this theorem we note how it is used to give the desired constructions.

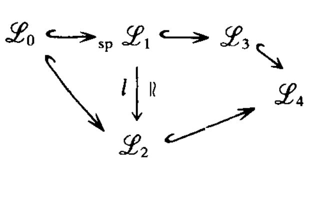

Diagram 4.2.

COROLLARY 4.2. *Every finite lattice $\mathscr{L}$ has a recursive sequential table Φ.*

PROOF. The trivial table consisting of just the map $0 \mapsto 0$ for each Φ_i is clearly a table for $\mathscr{L}_0 = \{0\}$. Apply the theorem with $\mathscr{L}_0 = \{0\} = \mathscr{L}_1 = \mathscr{L}_3$ and $\mathscr{L} = \mathscr{L}_4$. □

LEMMA 2.9. *If Θ is a recursive sequential table for $\mathscr{L}$ and $\mathscr{L}'$ is a finite extension of $\mathscr{L}$ then there is a recursive sequential table Φ for $\mathscr{L}'$ which refines Θ.*

PROOF. Apply the theorem by setting $\mathscr{L}_0 = \{0\} = \mathscr{L}_1$, $\mathscr{L}_3 = \mathscr{L}$ and $\mathscr{L}_4 = \mathscr{L}'$. Φ is then the required table and k shows that it refines Θ. □

LEMMA 3.9. *If $\mathscr{P}$ is appropriate to $\mathscr{L}$ and $\mathscr{L}$ is saturated then the sets D_{5,L_0,L_1} and $D'_{5,I,R,g,f}$ are dense in $\mathscr{P}$.*

PROOF. D_{5,L_0,L_1}: We are given $P \in \mathscr{P}$ and $L_0 \subseteq_{sp} L_1(\mathscr{L})$. By Lemma 3.4 there is an $L \supseteq L_P$ with $L_0 \subseteq_{sp} L_1(L)$. The proof of Lemma 2.11 for this L then gives a $Q \leqq P$ with $L_Q = L$ and so $Q \in D_{5,L_0,L_1}$.

$D'_{5,I,R,g,f}$: We may assume that the given $P \in \mathscr{P}$ refines R. As in the proof of Lemma 3.10(f) we may choose an $\mathscr{L}_4$ containing L_P and a realization L'_1 of $\underline{L}'_1$ given by an h extending f on $\underline{L}'_0$ such that $h[\underline{L}'_0] \subseteq_{sp} h[\underline{L}'_1](\mathscr{L})$ and such that any element x of $hj[\underline{L}_1]$ which is below any element $y \in L_P$ is in fact in $h[\underline{L}'_0] \subseteq L_P$. We now apply the theorem with $\mathscr{L}_0 = g[\underline{L}_0] \subseteq_{sp} g[\underline{L}_1] = \mathscr{L}_1$, $\mathscr{L}_3 = L_P$, $\Theta = \Theta_P$, $\mathscr{L}_2 = hj[\underline{L}_1]$ and l of the theorem induced by the j of the dense set in the obvious way: $l(g(\underline{x})) = hj(\underline{x})$. Let Φ, k, g and F be as in the conclusion of the theorem. We define our required $Q \leqq P$ by first setting $L_Q = \mathscr{L}_4$ and $\Theta_Q = \Phi$. The trees $T_{Q,x}$ in Q are defined by cases:

(a) $x \in L_P$. Set $T_{Q,x}(\emptyset) = T_{P,x}(0^{k(0)})$. [Note that as $\Phi \restriction L_P$ refines Θ any $\Theta \restriction x$ string is a $\Phi \restriction x$ string for $x \in L_P$.] Suppose by induction that $T_{Q,x}(\sigma)$ is defined for lth $\sigma = i$ so that there is a τ of length $k(i)$ such that $T_{Q,x}(\sigma) = T_{P,x}(\tau)$. We wish to define $T_{Q,x}(\sigma * n)$ for $n \in \Phi_i \restriction x$. As k shows that Φ refines Θ, $n \in \Theta_{k(i)} \restriction x$ and so we may set $T_{Q,x}(\sigma * n) = T_{P,x}(\tau * n^{k(i+1)-k(i)})$ and continue the induction.

(b) For $x \notin \mathscr{L}_2 = hj[\underline{L}_1]$ (and $x \notin L_P$), let $T_{Q,x}$ be the $\Phi \restriction x$ identity tree.

(c) For $l(x) \in \mathscr{L}_2$ we build a subtree of $T_{P,x}$. We begin by setting $T_{Q,l(x)}(\emptyset) = T_{P,x}(0^{k(0)})$. Again suppose inductively that, for $\text{lth}(\sigma) = i$, $T_{Q,l(x)}(\sigma)$ is defined so that there is a τ of length $k(i)$ with $T_{Q,l(x)}(\sigma) = T_{P,x}(\tau)$. We now wish to define $T_{Q,l(x)}(\sigma * n)$ for $n \in \Phi_i \restriction l(x)$, i.e., $n = \alpha(l(x))$ for some $\alpha \in \Phi_i$. By conclusion (iv) of the theorem $Fg\alpha \in \Theta_{k(i)}$ and so we may set

$$T_{Q,l(x)}(\sigma * n) = T_{P,x}(\tau * (Fg\alpha(x)^{k(i+1)-k(i)})$$

and continue the induction. Note that this definition depends only on $n = \alpha(l(x))$ and not on the choice of α since for $\beta \in \Phi_i$, $\beta(l(x)) = \alpha(l(x)) \Leftrightarrow Fg\beta(x) = Fg\alpha(x)$ as $Fg\alpha(x) = F_x(\alpha(l(x))$ and F_x is 1–1.

One should also note that the directions in cases (a) and (c) give the same results for $x \in L_P \cap \mathcal{L}_2 = \mathcal{L}_0$ since for such x, $l(x) = x$ and $Fg\alpha(x) = \alpha(x)$ for any $\alpha \in \Phi_i$ by conclusions (i) and (ii) of the theorem.

The maps between $[T_{Q,x}]$ and $[T_{Q,y}]$ required in the definition of Q are just those induced by Φ in the usual way. As Φ refines Ψ the situation is exactly as in the proof of Lemma 2.1 and the maps in Q for $y < x$ in L_P are just the restrictions of those in P. Thus $Q \leqq P$.

All that remains is to verify that $\phi(Q) \leqq R \restriction \mathcal{L}_1$ where ϕ is specified as in the definition of the dense set by $hjg^{-1}(x) \mapsto x$ for $x \in L_1 = g[\underline{L}_1]$ and $\operatorname{dom} \phi = hj[\underline{L}_1]$. This map is however precisely l^{-1} on $\mathcal{L}_2$. Of course the tree $T_{l(Q),x}$ is just $T_{Q,l(x)}$ which was defined as a subtree of $T_{P,x} \subseteq T_{R,x}$ as required. As for the maps, consider any $y < x$ in $\mathcal{L}_1$ and suppose that $S_x \in [T_{Q,l(x)}]$ is mapped to $S_y \in [T_{Q,l(y)}]$ by the maps $F_{Q,l(x),l(y)}$. We must show that $F_{R,x,y}(S_x) = F_{P,x,y}(S_x) = F_{Q,x,y}(S_x) = S_y$. Recall, however, that if $S_x = T_{Q,l(x)}[h]$ then

$$S_y = T_{Q,l(y)}[f_{Q,l(x),l(y)}h].$$

Thus

$$S_x = T_{P,x}(0^{k(0)}) * \langle Fg\alpha_i(x)^{k(i+1)-k(i)} \mid i \in \omega \rangle$$

where $\alpha_i(l(x)) = h(i)$. If we apply $F_{P,x,y}$ we get

$$T_{P,y}(0^{k(0)}) * \langle Fg\alpha_i(y)^{k(i+1)-k(i)} \mid i \in \omega \rangle.$$

On the other hand $S_y = T_{Q,l(y)}[f_{Q,l(x),l(y)}h] = T_{Q,l(y)}[\langle \alpha_i(l(y)) \mid i \in \omega \rangle]$ where again $\alpha_i(l(x)) = h(i)$. By definition of $T_{Q,l(y)}$ this is just

$$T_{P,y}(0^{k(0)}) * \langle Fg\alpha_i(y)^{k(i+1)-k(i)} \mid i \in \omega \rangle$$

as required. □

Before diving into the proof of Theorem 4.1 we note a few useful facts and lemmas (with the notation as in the theorem).

FACT 4.3. If g satisfies the other conditions imposed in (i) then $g\Phi_i$ is automatically a positive table for $\mathcal{L}_4$. (Thus $\Phi_i \cup g\Phi_i$ is a table for $\mathcal{L}_4$.)

PROOF. We must verify clauses (i)–(iii) of Definition 2.1.

(i) As $0 \in \operatorname{dcl}_{\mathcal{L}_4} \mathcal{L}_1$, $g\alpha(0) = \alpha(\Pi_0(0) \vee l\Pi_1(0)) = \alpha(0) = 0$ for every $\alpha \in \Phi_i$.

(ii) Suppose $\alpha, \beta \in \Phi_i$, $x \preccurlyeq y \in \mathcal{L}_4$ and $g\alpha(y) = g\beta(y)$. As we may assume that $\alpha \neq \beta$ the conditions on g imply that $y \in \operatorname{dcl}_{\mathcal{L}_4} \mathcal{L}_1$. Thus $g\alpha(y) =$

$\alpha(\Pi_0(y) \vee l\Pi_1(y)) = \beta(\Pi_0(y)) \vee l\Pi_1(y)) = g\beta(y)$. As $x \leqslant y$, $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_1$ and so $g\alpha(x) = \alpha(\Pi_0(x) \vee l\Pi_1(x))$, $g\beta(x) = \beta(\Pi_0(x) \vee l\Pi_1(x))$. As $x \leqslant y$, $\Pi_i(x) \leqq \Pi_i(y)$ and so $l\Pi_1(x) \leqq l(\Pi_1(y))$ as well. Thus $\Pi_0(x) \vee l(\Pi_1(x)) \leqq \Pi_0(y) \vee l\Pi_1(y))$. As Φ_i itself satisfies (ii) of Definition 2.1, $g\alpha(x) = \alpha(\Pi_0(x) \vee l\Pi_1(x)) = \beta(\Pi_0(x) \vee l\Pi_1(x)) = g\beta(x)$.

(iii) As above $g\alpha(x) = g\beta(x)$, $g\alpha(y) = g\beta(y)$ and $z = x \vee y$ imply that $x, y, z \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_1$ and so $\alpha(\Pi_0(z) \vee l\Pi_1(z)) = g\alpha(z)$ and, similarly for β, x and y. Now as $\mathscr{L}_0 \subseteq_{\mathrm{sp}} \mathscr{L}_1(\mathscr{L}_4)$ the associated generation process produces $\Pi_0(z)$, $\Pi_1(z)$ from $\{\Pi_i(x), \Pi_i(y)\}$ entirely inside $\mathscr{L}_4$. If we apply the same process to $\mathscr{L}_0 \subseteq \mathscr{L}_2$ in place of $\mathscr{L}_0 \subseteq \mathscr{L}_1$ to $\{\Pi_0(x), l\Pi_1(x), \Pi_0(g), l\Pi(y)\}$ we of course get $\Pi_0(z)$, $l\Pi_1(z)$. By clauses (ii) and (iii) of Definition 2.1 the generating process preserves equality of values for different $\alpha, \beta \in \Phi_i$. Now $g\alpha \equiv g\beta$ modulo x and $y, \alpha \equiv \beta$ mod $\Pi_0(x)$, $l\Pi_1(x)$, $\Pi_0(y)$ and $l\Pi_1(y)$. Thus $\alpha \equiv \beta$ modulo any element generated in this process and so in particular $\Pi_0(z)$ and $l\Pi_1(z)$. Thus $\alpha \equiv \beta \bmod(\Pi_0(z) \vee l\Pi_1(z))$ as required. □

FACT 4.4. The conditions on g and F imply that $g\Phi_i \restriction \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_1$ and so $Fg\Phi_i \restriction \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_1$ are uniquely determined by Φ_i, l and F_x for $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_1$ and in fact $\Phi_i \subseteq_a \Phi_i \cup Fg\Phi_i$.

PROOF. The uniqueness is clear. By Fact 4.3 $\Phi_i \cup Fg\Phi_i$ is a table for $\mathscr{L}_4$. We must check admissibility. Consider any $Fg\alpha$ for $\alpha \in \Phi_i$. We claim that α is the required witness in Φ_i: If $\gamma \in \Phi_i$ and $Fg\alpha(x) = \gamma(x)$ then $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$ by the requirements on F and g. Thus $Fg\alpha(x) = \alpha(x)$ as required. □

FACT 4.5. (iii) $\Rightarrow$ (iv) & (v).

PROOF. Suppose $\alpha \in \Theta_{k(i)}$ and has a witness for admissibility $\beta \in \Phi_i \cup Fg\Phi_i$, i.e.,

$$\forall \gamma \in \Phi_i \cup Fg\Phi_i \ \forall x \in \mathscr{L}_3[\alpha \equiv_x \gamma \rightarrow \alpha \equiv_x \beta].$$

β clearly witnesses (iv) or (v) according to which of Φ_i, $Fg\Phi_i$ it belongs to. Suppose $\beta \in \Phi_i$ and we have $\gamma \in \Phi_i$ and x with $Fg\gamma \equiv_x \alpha$. Then $Fg\gamma \equiv_x \beta$ by admissibility. The conditions on F then imply that $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$ and so $Fg\gamma(x) = \gamma(x) = Fg\beta(x) = \beta(x)$. Thus $Fg\beta$ is the required witness for (v). On the other hand if $\beta \in Fg\Phi_i$, say, $\beta = Fg\delta$. If we are given $\gamma \in \Phi_i$ with $\alpha \equiv_x \gamma$ then $Fg\delta \equiv_x \gamma$ and again $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$. Thus $\delta(x) = Fg\delta(x) = \gamma(x) = \alpha(x)$ and so δ is the required witness for (iv). □

Now to build Φ_i so that Φ is a sequential table we must guarantee that the elements (interpolants) required by Definition 2.2b (iii) and (iv) for elements in

Φ_i exist in Φ_{i+1}. To this end we cite a definition and result from Lerman [10, appendix B, 3.11 and 3.12] with the proviso that one reverts to the treatment of (iv) in Lerman [9] as explained above for Definition 2.1:

DEFINITION 4.6. A finite table Ψ^* extending one Ψ (for some $\mathscr{L}$) is a *type* 1 *extension* of Ψ if $\Psi \subseteq_a \Psi^*$ and the requirements of Definition 2.2b (iii) and (iv) hold for Ψ, Ψ^* in place of Θ_i, Θ_{i+1}.

LEMMA 4.7. *Every finite table Ψ for (a lattice $\mathscr{L}$) has a type* 1 *extension.* □

The key new lemma which allows us to build Φ to satisfy (iii) of the theorem (in addition to being a recursive extendible sequential table for $\mathscr{L}_4$) is the following.

LEMMA 4.8. *If we have constructed (by induction) Φ_i and have defined g on Φ_i and F on $g\Phi_i$ so as to satisfy* (i)–(iii) *and we are given a Ψ such that $\Phi_i \hookrightarrow_a \Phi_i \cup \Psi$ (a table for $\mathscr{L}_4$) then we can find $H : \Psi \simeq \Psi^*$ with $H_x(\alpha(x)) = \alpha(x)$ if $\exists \beta \in \Phi_i \cup Fg\Phi_i[\beta(x) = \alpha(x)]$ and otherwise $H_x(\alpha(x)) > j$ for any specified j (so that $\Phi_i \hookrightarrow_a \Phi_i \in \Psi^*$), extensions of F and g and a $k > k(i)$ so that* (i)–(iii) *remain satisfied for these extensions, in particular*

$$(\Phi_i \cup \Psi^*) \cup Fg(\Phi_i \cup \Psi^*) \restriction \mathscr{L}_3 \hookrightarrow_a \Theta_k.$$

PROOF. First note that for any table $\Phi_i \hookrightarrow_a \Phi_i \cup \Psi$ and H as described, $\Phi_i \hookrightarrow_a \Phi_i \cup \Psi^*$: Consider any $\alpha \in \Phi_i \cup \Psi^*$. If $\alpha \in \Phi_i$ it is its own witness. If $\alpha \in \Psi^*$ then $\alpha = H\delta$ for some $\delta \in \Psi$ which has a witness $\beta \in \Phi_i$. If $\gamma \in \Phi_i$, $x \in \mathscr{L}_4$ and $\alpha \equiv_x \gamma$ then by definition of H, $\alpha(x) = H\delta(x) = \delta(x) = \gamma(x)$. As β is a witness for δ, $\delta(x) = \beta(x)$. Thus β is also a witness for α. Now extend g to g^* on $\Phi_i \cup \Psi$ as specified by (i) choosing elements not in the ranges of Φ_i, Ψ, $g\Phi_i$ or $Fg\Phi_i$ when new elements are called for. Similary let F^* extend F (i.e., the finite amount defined so far) as required in (ii). Again new elements are chosen from those not yet appearing in the construction. By Fact 4.3 $g^*(\Phi_i \cup \Psi)$ and so $F^*(\Phi_i \cup \Psi)$ are positive tables for $\mathscr{L}_4$ and so $\Phi_i \cup \Psi \cup F^*g^*(\Phi_i \cup \Psi)$ is a table for $\mathscr{L}_4$.

CLAIM 1. $\Phi_i \cup F^*g^*(\Phi_i) \subseteq_a (\Phi_i \cup \Psi) \cup F^*g^*(\Phi_i \cup \Psi)$.

PROOF. Consider first any $\alpha \in \Phi_i \cup \Psi$. It has a witness $\beta \in \Phi_i$ to $\Phi_i \hookrightarrow_a \Phi_i \cup \Psi$. It is also the witness we require: Consider any $\gamma \in \Phi_i \cup \Psi \cup F^*g^*(\Phi_i \cup \Psi)$. If $\gamma \in \Phi_i \cup \Psi$ we are done. If $\gamma = F^*g^*\delta$ for $\delta \in \Phi_i \cup \Psi$ then as $\gamma(x) = \alpha(x)$, $\delta \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$ and $\gamma(x) = F^*g^*\delta(x) = g^*\delta(x) = \delta(x)$. Thus $\beta(x) = \delta(x) = \gamma(x)$ as required.

Next consider $F^*g^*\alpha$ for $\alpha \in \Phi_i \cup \Psi$. We claim that $F^*g^*\beta$ is the required witness where β is the one for α in $\Phi_i \subseteq_a \Phi_i \cup \Psi$. Consider any γ and x with $F^*g^*\alpha(x) = \gamma(x)$. If $\gamma \in \Phi_i \cup \Psi$ then $x \in \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_0$ and $F^*g^*\alpha(x) = g^*\alpha(x) = \alpha(x) = \gamma(x) = \beta(x) = g^*\beta(x) = F^*g^*\beta(x)$ as required. On the other hand if $\gamma = F^*g^*\delta$ for some $\delta \in \Phi_i \cup \Psi$ then $F^*g^*\delta(x) = F^*g^*\alpha(x)$ and so by the requirements on F^*, $g^*\delta(x) = g^*\alpha(x)$. Thus by the requirements on g^*, $x \in \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_1$ and $g^*\delta(x) = \delta(\Pi_0(x) \vee l\Pi_1(x)) = \alpha(\Pi_0(x) \vee l\Pi_1(x)) = g^*\alpha(x)$. As β is a witness for α, $\alpha(\Pi_0(x) \vee l\Pi_1(x)) = \beta(\Pi_0(x) \vee l\Pi_1(x)) = g^*\beta(x) = F^*g^*\beta(x) = F^*g^*\delta(x) = \gamma(x)$ as required for $F^*g^*\beta$ to be the desired witness.

□

We can now apply (vi) of the definition of an extendible table (2.3(b)) to $\Theta_{k(i)}$ with $j' > j$ larger than any element used so far to get a $k > k(i)$ and an isomorphism P as there described to yield the following diagram:

$$\begin{array}{lcr}
\Sigma = \Phi_i \cup F^*g^*\Phi_i \longrightarrow \Sigma \restriction \mathcal{L}_3 \hookrightarrow_a & & \Theta_{k(i)} \\
\quad a \downarrow & & \downarrow a \\
\Phi_i \cup F^*g^*\Phi_i \cup \Psi \cup F^*g^*\Psi & & \Theta'_{j'} \\
\quad P \downarrow \wr & & \downarrow a \\
\Upsilon = \Phi_i \cup F^*g^*\Phi_i \cup P\Psi \cup PF^*g^*\Psi & \longrightarrow T \restriction \mathcal{L}_3 \hookrightarrow_a & \Theta_k
\end{array}$$

It now suffices to show that we can define $H : \Psi \xrightarrow{\sim} \Psi^*$ and extensions of F and g such that $\Psi^* \cup Fg\Psi^* = P\Psi \cup PF^*g^*\Psi$. We first claim that we can set $H = P$. The requirements on P in (vi) are precisely those needed for H in the theorem. Thus the final claim is that we can define acceptable extensions F^+ and g^+ of F and g so that $PF^*g^*\Psi = F^+g^+\Psi^* = F^+g^+P\Psi$. Suppose then that $\alpha \in \Psi - \Phi_i$. We must define $g^+P\alpha$. If $x \notin \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_1$ then we can let $g^+P\alpha(x)$ be any new distinct element and then set $F^+g^+P\alpha(x)$ to be $PF^*g^*\alpha(x)$. (As g^* suitably extends g, $g^*\alpha(x)$ and so $F^*g^*\alpha(x)$ are not mentioned in $Fg\Phi_i \cup \Phi_i$. The definition of P then makes $PF^*g^*\alpha(x) > j$ and so a new number eligible to be $F^+g^+P\alpha(x)$.) Thus for $x \notin \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_1$ we have suitably defined F_x^+ and $g^+P\alpha(x)$ for $\alpha \in \Psi$.

Next consider an $x \in \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_1$ so that $x = \Pi_0(x) \vee \Pi_1(x)$. We are required to set $g^+P\alpha(x) = (P\alpha)(\Pi_0(x) \vee l\Pi_1(x))$. Thus we must set

$$F_x^+(P\alpha(\Pi_0(x)) \vee l\Pi_1(x)) = PF^*g^*\alpha(x) = P_x F_x^*(\alpha(\Pi_0(x) \vee l\Pi_1(x))).$$

We must now verify that F_x^+ suitably extends F_x. The first concern is that if $x \in \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_0$ then $F_x = \mathrm{id}$. In this case, however, $\Pi_0(x) \vee l\Pi_1(x) = x$ and so $g^+P\alpha(x) = P\alpha(x)$ and $PF^*g^*\alpha(x) = PF^*\alpha(x) = P\alpha(x)$ as $F_x^* = \mathrm{id}$ as well. The only other concern is that, for $x \in \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_1 - \mathrm{dcl}_{\mathcal{L}_4} \mathcal{L}_0$, F_x^+ extend F_x, i.e.,

$F_x((P\alpha)(\Pi_0(x) \vee l\Pi_1(x)))$ may already be defined. This can happen, however, only if $\exists \beta \in \Phi_i$ with $\beta(x) = (P\alpha)(\Pi_0(x) \vee l\Pi_1(x))$. By our choice of P, however, this can occur only if

$$\exists \delta \in (\Phi_i \cup F^+ g^+ \Phi_i)[\delta(\Pi_0(x) \vee l\Pi_1(x)) = \alpha(\Pi_0(x) \vee l\Pi_1(x))]$$

in which case

$$(P\alpha)(\Pi_0(x) \vee l\Pi_1(x)) = \alpha(\Pi_0(x) \vee l\Pi_1(x)).$$

Thus $F_x(\alpha(\Pi_0(x) \vee l\Pi_1(x)))$ is defined and so equal to $F^*_x(\alpha(\Pi_0(x) \vee l\Pi_1(x)))$ as $F^* \supset F$. If $\delta \in \Phi_i$ then

$$F^*_x(\alpha(\Pi_0(x) \vee l\Pi_1(x)) = F^*_x(\delta(\Pi_0(x) \vee l\Pi_1(x))) = F^* g^* \delta(x)$$

and so $P_x(F^* g^* \delta(x)) = F^* g^* \delta(x) = F^*_x(\alpha(\Pi_0(x) \vee l\Pi_1(x)))$ as required. On the other hand if $\delta = F^* g^* \gamma$ for some $\gamma \in \Phi_i$ then we could not have $\delta(\Pi_0(x) \vee l\Pi_1(x)) = \beta(x)$. The point here is that as $x \notin \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$, $l\Pi_1(x) \notin \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_1$ and so $g^*\gamma$ and so $F^* g^* \gamma$ at $\Pi_0(x) \vee l\Pi_1(x)$ are by choice of g^*, F^* elements not mentioned in Φ_i. Thus $F^+_x \supseteq F_x$ as required. □

Proof of Theorem 4.1. We construct Φ_i and define F and g on the appropriate domains by induction.

Step 0. Let Ψ be any finite table for $\mathscr{L}_4 \supseteq \mathscr{L}_3$. Apply (v) of the definition of extendibility (2.3(b)) to Θ to get $\Psi \simeq \Psi^* \to \Psi^* \restriction \mathscr{L}_3 \hookrightarrow \Theta_j$. Define g^* on Ψ^* and F^* on $g^*\Psi^*$ as required in (i) and (ii) of the theorem with new elements chosen outside of Θ_j as well as Ψ^*. By Fact 4.4 $\Psi^* \cup F^* g^* \Psi^*$ is a table for $\mathscr{L}_4$ admissibly extending Ψ^*. We can now apply (vi) of Definition 2.3(b) to get

$$\begin{array}{ccccc}
\Psi^* & \longrightarrow & \Psi^* \restriction \mathscr{L}_3 & \hookrightarrow_a & \Theta_j \\
\downarrow & & & & \downarrow a \\
\Psi^* \cup F^* g^* \Psi^* & & & & \\
\downarrow \wr & & & & \\
\Sigma = \Psi^* \cup PF^* g^* \Psi^* & \longrightarrow & \Sigma \restriction \mathscr{L}_3 & \hookrightarrow & \Theta_k
\end{array}$$

We can now set $\Phi_0 = \Psi^*$, $g = g^*$ and $F = PF^*$. Thus $\Phi_0 \cup Fg\Phi_0 \restriction \mathscr{L}_3 \hookrightarrow_a \Theta_k$ and we may set $k(0) = k$ to begin the construction. The only point to verify is that F satisfies the requirements of the theorem: If $x \in \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$ and $\alpha \in \Phi_0 = \Psi^*$ then $g\alpha(x) = \alpha(x)$ and $F_x(g\alpha)(x) = P_x F^*_x(g\alpha)(x) = P_x\alpha(x) = \alpha(x)$ as $P\alpha = \alpha$ for $\alpha \in \Psi^*$. On the other hand if $x \notin \mathrm{dcl}_{\mathscr{L}_4} \mathscr{L}_0$ then $F^* g^* \alpha(x)$ and so the $PF^* g^* \alpha(x) = Fg\alpha(x)$ are now distinct elements as required.

We now list all possible instances of (v) and (vi) of the definition of extendibility for $\mathscr{L}_4$, Φ and satisfy the nth ones at stage $2n+1$, $2n+2$ respectively. In either case we first make sure we get a type 1 extension.

Step $i+1$. By Lemma 4.7, Φ_i has a type 1 extension $\Phi_i^f \cup \Psi$. By Lemma 4.8 we can find $k > k(i)$, $H : \Psi^* \simeq \Psi$ and extensions for F and g so that $\Phi_i \subseteq_a \Phi_i \cup \Psi^*$ and

$$(\Phi_i \cup \Psi^*) \cup Fg(\Phi_i \cup \Psi^*) \restriction \mathscr{L}_3 \hookrightarrow_a \Theta_k.$$

As $H_x(\alpha(x)) = \alpha(x)$ if $\exists \beta \in \Phi_i[\beta(x) = \alpha(x)]$, it is clear that $\Phi_i \cup \Psi^*$ is also a type 1 extension of Φ_i as is any admissible extension of it. For notational convenience let $\Phi_i \cup \Psi^* = \Phi'$. We now divide into cases by the parity of i.

$i = 2n$. We must guarantee that Φ satisfies the nth instance of (v) of Definition 2.3(b). Suppose it is given by a table Ψ for $\mathscr{L}' \supseteq \mathscr{L}_4$. We must build Φ_{i+1} an admissible extension of Φ', an isomorphism $P : \psi \xrightarrow{\sim} \Psi^* \rightarrow \Psi^* \restriction \mathscr{L}_4 \subseteq_a \Phi_{i+1}$, and extensions of F and g as required.

We begin by choosing $\Psi' \simeq \Psi$ with all elements in the range of Ψ' new (i.e., not mentioned in Φ', $Fg\Phi'$ or Θ_k). Thus $\Phi' \subseteq_a \Phi' \cup \Psi' \restriction \mathscr{L}_4$. We can now apply Lemma 4.8 to get a $k' > k$, an $H : \Psi' \restriction \mathscr{L}_4 \rightarrow \Psi^*$ with $\Phi' \subseteq_a \Phi \cup \Psi^*$ and

$$(\varphi' \cup \Psi^* \cup Fg(\Phi' \cup \Psi^*)) \restriction \mathscr{L}_3 \subseteq_a \Theta_k.$$

We can clearly extend H by setting $H_x = \mathrm{id}$ for $x \notin \mathscr{L}_4$ so that $H\Psi' \simeq \Psi' \simeq \Psi$. Thus we have

$$\begin{array}{c} \Psi \\ \downarrow \\ H\Psi' \rightarrow H\Psi' \restriction \mathscr{L}_4 = \Psi^* \subseteq_a \Phi' \cup \Psi^*. \end{array}$$

We may now set $\Phi_{i+1} = \Phi' \cup \Psi^*$, $k(i+1) = k'$ and extend F and g as specified by Lemma 4.8 to satisfy the nth instance of (v) and keep the induction going.

$i = 2n+1$. We must guarantee that Φ_{i+1} satisfies the nth instance of (vi). Suppose it is given by a table Ψ for $\mathscr{L}' \supseteq \mathscr{L}_4$ with $\Psi \restriction \mathscr{L}_4 \subseteq_a \Phi_{i'} \subseteq_a \Phi_i$ (some $i' \leqq i$), a Ψ^* admissibly extending Ψ and a $j \leqq i$. It suffices to build $\Phi_{i+1} \supseteq_a \Phi'$ and a $P : \Psi^* \simeq \Psi^*$ with $\Psi^+ \restriction \mathscr{L}_4 \subseteq_a \Phi_{i+1}$ and to extend F and g appropriately to satisfy (i)–(iii).

We begin by defining P^* on Ψ^* so that $P_x^*\alpha(x) = \alpha(x)$ if $\exists \beta \in \Psi[\beta(x) = \alpha(x)]$ and otherwise P_x^* sends everything to new larger elements. We claim that $\Phi' \subseteq_a \Phi' \cup P^*\Psi^* \restriction \mathscr{L}_4$. For any $P^*\alpha^*$ with $\alpha^* \in \Psi^* \restriction \mathscr{L}_4$, α^* has a witness $\alpha \in \Psi$ to $\Psi \subseteq_a \Psi^*$ and α has one $\beta \in \Phi'$ to $\Psi \restriction \mathscr{L}_4 \subseteq_a \Phi'$. β is the required witness for $P^*\alpha^*$. Consider any $\gamma \in \Phi'$, $x \in \mathscr{L}_4$ with $\gamma(x) = P^*\alpha^*(x)$. By the choice of P^*, $\exists \delta \in \Psi[\delta(x) = \alpha^*(x)]$ and $P^*\alpha^*(x) = \alpha^*(x) =$

$\delta(x)=\gamma(x)$. By the choice of α, $\alpha(x)=\delta(x)\,(=\gamma(x))$ and so by the choice of β, $\beta(x)=\gamma(x)=P^*\alpha^*(x)$ as required.

We can now apply Lemma 4.8 (for j of this instance) to get $H: P^*\Psi^*\restriction \mathcal{L}_4 \tilde{\rightarrow} \Psi^+\restriction \mathcal{L}_4$ where $\Psi^+=HP^*\Psi^*$ once one extends H by setting $H_x=\mathrm{id}$ for $x\notin \mathcal{L}_4$. This gives us $\Phi' \subseteq_a \Phi' \cup \Psi^+\restriction \mathcal{L}_4$ and suitable extensions of F and g such that (i)–(iii) are satisfied for $\Phi_{i+1}=\Phi'\cup\Psi^+\restriction \mathcal{L}_4$'with some suitable $k'>k$. We can then set $k(i+1)=k'$. We have thus also satisfied (vi) by setting $P=HP^*:\Psi^*\rightarrow\Psi^+$ as long as P satisfies the conditions of (vi) and $\Psi^+\restriction\mathcal{L}_4\subseteq_a \Phi_{i+1}$. As for the first point, if $\alpha\in\Psi$ then $P^*\alpha=\alpha\in P^*\Psi^*$ but as $\Psi\restriction\mathcal{L}_4\subseteq\Phi_i\subseteq\Phi'$, $H_x(\alpha(x))=\alpha(x)$ for $x\in\mathcal{L}_4$ while for $x\notin\mathcal{L}_4$, $H_x=\mathrm{id}$. Thus $HP^*\alpha=\alpha$ for $\alpha\in\Psi$. On the other hand if $n\notin\Psi\restriction x$ then $P^*_x(n)\notin(\Phi'\cup Fg\Phi')\restriction x$ and so $H_xP^*_x(n)>j$ as required. Finally to see that $\Psi^+\restriction\mathcal{L}_4\subseteq_a\Phi_{i+1}=\Phi'\cup\Psi^+\restriction\mathcal{L}_4$ consider any $\alpha\in\Phi'$. As $\Psi^+\restriction\mathcal{L}_4\supseteq\Psi\restriction\mathcal{L}_4\subseteq\Phi'$ we may choose a witness $\beta\in\Psi$. We claim β works for $\Psi^+\restriction\mathcal{L}_4$ as well. If $\gamma\in\Psi^+$, $x\in\mathcal{L}_4$ and $\gamma(x)=\alpha(x)$ then by choice of P^*, $\exists\delta\in\Psi$ with $\gamma(x)=\delta(x)=\alpha(x)$. Now by choice of β, $\beta(x)=\delta(x)=\alpha(x)$ as required. □

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